Hierarchy and fusion 3: Fusions with ILC

Lorenzo Sadun

University of Texas

September 20, 2017

Outline

1. What does ILC look like?

Outline

1. What does ILC look like?
2. Topological considerations

Outline

1. What does ILC look like?
2. Topological considerations
3. ILC fusions

Outline

1. What does ILC look like?
2. Topological considerations
3. ILC fusions
4. Invariant Measures

Outline

1. What does ILC look like?
2. Topological considerations
3. ILC fusions
4. Invariant Measures
5. Complexity

Outline

1. What does ILC look like?
2. Topological considerations
3. ILC fusions
4. Invariant Measures
5. Complexity
6. Tiling with infinitely many sizes

Outline

1. What does ILC look like?
2. Topological considerations
3. ILC fusions
4. Invariant Measures
5. Complexity
6. Tiling with infinitely many sizes
7. Summary

Table of Contents

1. What does ILC look like?
2. Topological considerations
3. ILC fusions
4. Invariant Measures
5. Complexity
6. Tiling with infinitely many sizes
7. Summary

What is FLC?

- Finitely many tile types (prototiles) $\{t_i\}$.

What is FLC?

- Finitely many tile types (prototiles) $\{t_i\}$.
- Finitely many ways to have two tiles meet.

What is FLC?

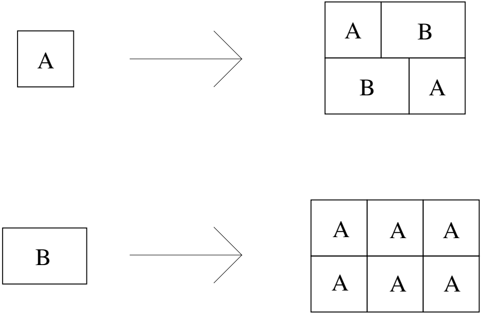
- Finitely many tile types (prototiles) $\{t_i\}$.
- Finitely many ways to have two tiles meet.
- Finitely many patches of size r, up to translation.

What is FLC?

- Finitely many tile types (prototiles) $\{t_i\}$.
- Finitely many ways to have two tiles meet.
- Finitely many patches of size r, up to translation.
- Excludes many interesting patterns.

Rotational ILC

Shear ILC



Combinatorial ILC

- Dyadic solenoid. Tiles $A_1, \ldots, A_\infty$.
- All tiles have length 1.

Combinatorial ILC

- Dyadic solenoid. Tiles $A_1, \ldots, A_\infty$.
- All tiles have length 1.
- Put A_1 in every other other slot:
  $\ldots A_1 \quad A_1 \quad A_1 \quad A_1 \quad A_1 \quad \ldots$

Combinatorial ILC

- Dyadic solenoid. Tiles $A_1, \ldots, A_\infty$.
- All tiles have length 1.
- Put A_1 in every other other slot:
  $\ldots A_1 \quad A_1 \quad A_1 \quad A_1 \quad A_1 \quad \ldots$
- Put A_2 in every other remaining slot:
  $\ldots A_1 A_2 A_1 \quad A_1 A_2 A_1 \quad A_1 A_2 \ldots$

Combinatorial ILC

- Dyadic solenoid. Tiles $A_1, \ldots, A_\infty$.
- All tiles have length 1.
- Put A_1 in every other other slot:
  $\ldots A_1 \quad A_1 \quad A_1 \quad A_1 \quad A_1 \quad \ldots$
- Put A_2 in every other remaining slot:
  $\ldots A_1 A_2 A_1 \quad A_1 A_2 A_1 \quad A_1 A_2 \ldots$
- Put A_3 in every other remaining slot:
  $\ldots A_1 A_2 A_1 A_3 A_1 A_2 A_1 \quad A_1 A_2 \ldots$

Combinatorial ILC

- Dyadic solenoid. Tiles $A_1, \ldots, A_\infty$.
- All tiles have length 1.
- Put A_1 in every other other slot:
  $\ldots A_1 \quad A_1 \quad A_1 \quad A_1 \quad A_1 \quad \ldots$
- Put A_2 in every other remaining slot:
  $\ldots A_1 A_2 A_1 \quad A_1 A_2 A_1 \quad A_1 A_2 \ldots$
- Put A_3 in every other remaining slot:
  $\ldots A_1 A_2 A_1 A_3 A_1 A_2 A_1 \quad A_1 A_2 \ldots$
- Lather, rinse, repeat infinitely many times.

Combinatorial ILC

- Dyadic solenoid. Tiles $A_1, \ldots, A_\infty$.
- All tiles have length 1.
- Put A_1 in every other other slot:
  $\ldots A_1 \quad A_1 \quad A_1 \quad A_1 \quad A_1 \quad \ldots$
- Put A_2 in every other remaining slot:
  $\ldots A_1 A_2 A_1 \quad A_1 A_2 A_1 \quad A_1 A_2 \ldots$
- Put A_3 in every other remaining slot:
  $\ldots A_1 A_2 A_1 A_3 A_1 A_2 A_1 \quad A_1 A_2 \ldots$
- Lather, rinse, repeat infinitely many times.
- If a slot is still empty, put A_∞ there.

Thermal ILC

- Start with point set associated with FLC tiling.

Thermal ILC

- Start with point set associated with FLC tiling.
- Move each point by a small random amount.

Thermal ILC

- Start with point set associated with FLC tiling.
- Move each point by a small random amount.
- Build tiling from new points (e.g. from Voronoi cells).

Table of Contents

1. What does ILC look like?
2. Topological considerations
3. ILC fusions
4. Invariant Measures
5. Complexity
6. Tiling with infinitely many sizes
7. Summary

Is FLC the same as compactness?

- Usual tiling metric: T and T' are ϵ-close if T and T' agree on $B_{1/\epsilon}$, up to rigid ϵ-translation.

Is FLC the same as compactness?

- Usual tiling metric: T and T' are ϵ-close if T and T' agree on $B_{1/\epsilon}$, up to rigid ϵ-translation.
- In that metric topology, T has FLC $\Leftrightarrow$ Ω_T is compact.

Is FLC the same as compactness?

- Usual tiling metric: T and T' are ϵ-close if T and T' agree on $B_{1/\epsilon}$, up to rigid ϵ-translation.
- In that metric topology, T has FLC $\Leftrightarrow$ Ω_T is compact.
- Abandon compactness or use different topology?

Is FLC the same as compactness?

- Usual tiling metric: T and T' are ϵ-close if T and T' agree on $B_{1/\epsilon}$, up to rigid ϵ-translation.
- In that metric topology, T has FLC $\Leftrightarrow$ Ω_T is compact.
- Abandon compactness or use different topology?
- Use different topology!

Space of tile labels

- In FLC world, {tile labels} is finite.

Space of tile labels

- In FLC world, $\{$tile labels$\}$ is finite.
- In ILC world, $\{$tile labels$\}$ is *compact*.

Space of tile labels

- In FLC world, {tile labels} is finite.
- In ILC world, {tile labels} is *compact*.
  - For pinwheel, {labels} $= O(2)$.

Space of tile labels

- In FLC world, $\{$tile labels$\}$ is finite.
- In ILC world, $\{$tile labels$\}$ is *compact*.
  - For pinwheel, $\{$labels$\} = O(2)$.
  - For dyadic solenoid, $\{$labels$\} =$ 1-point compactification of $\mathbb{N}$.

Space of tile labels

- In FLC world, $\{\text{tile labels}\}$ is finite.
- In ILC world, $\{\text{tile labels}\}$ is *compact*.
    - For pinwheel, $\{\text{labels}\} = O(2)$.
    - For dyadic solenoid, $\{\text{labels}\} =$ 1-point compactification of $\mathbb{N}$.
- Geometry must be compatible: If $\text{label}_i \to \text{label}_\infty$, then (shape of t_i) $\to$ (shape of t_∞). (Hausdorff metric)

Topology of ILC tiling spaces

Tilings T and T' are ϵ-close if, on $B_{1/\epsilon}$,

Topology of ILC tiling spaces

Tilings T and T' are ϵ-close if, on $B_{1/\epsilon}$,

- There is a 1–1 correspondence between tiles in T and T',

Topology of ILC tiling spaces

Tilings T and T' are ϵ-close if, on $B_{1/\epsilon}$,

- There is a 1–1 correspondence between tiles in T and T',
- The labels of corresponding tiles are ϵ-close, and
- The locations of corresponding tiles are ϵ-close.

Topology of ILC tiling spaces

Tilings T and T' are ϵ-close if, on $B_{1/\epsilon}$,

- There is a 1–1 correspondence between tiles in T and T',
- The labels of corresponding tiles are ϵ-close, and
- The locations of corresponding tiles are ϵ-close.
- Can also speak of patches being ϵ-close.

Topology of ILC tiling spaces

Tilings T and T' are ϵ-close if, on $B_{1/\epsilon}$,

- There is a 1–1 correspondence between tiles in T and T',
- The labels of corresponding tiles are ϵ-close, and
- The locations of corresponding tiles are ϵ-close.
- Can also speak of patches being ϵ-close.
- In this topology, shears and rotations are continuous.
- In this topology, Ω_T is always compact.

Minimality and Repetitivity

- A dynamical system is *minimal* if every orbit is dense.

Minimality and Repetitivity

- A dynamical system is *minimal* if every orbit is dense.
- $T' \in \Omega_T$ iff every patch of T' is approximated arbitrarily well in T.

Minimality and Repetitivity

- A dynamical system is *minimal* if every orbit is dense.
- $T' \in \Omega_T$ iff every patch of T' is approximated arbitrarily well in T.
- For repetitivity, *do not* look at specific patches of diameter r. Look at open sets of patches.

Minimality and Repetitivity

- A dynamical system is *minimal* if every orbit is dense.
- $T' \in \Omega_T$ iff every patch of T' is approximated arbitrarily well in T.
- For repetitivity, *do not* look at specific patches of diameter r. Look at open sets of patches.
- Def: T is repetitive if, for each U, $\exists R$ s.t. every ball of radius R contains at least one patch from U.

Minimality and Repetitivity

- A dynamical system is *minimal* if every orbit is dense.
- $T' \in \Omega_T$ iff every patch of T' is approximated arbitrarily well in T.
- For repetitivity, *do not* look at specific patches of diameter r. Look at open sets of patches.
- Def: T is repetitive if, for each U, $\exists R$ s.t. every ball of radius R contains at least one patch from U.
- Ω_T is minimal iff T is repetitive.

Table of Contents

1 What does ILC look like?

2 Topological considerations

3 ILC fusions

4 Invariant Measures

5 Complexity

6 Tiling with infinitely many sizes

7 Summary

Ingredients

- **Compact** spaces $\mathcal{P}_n$ of n-supertiles $(n = 0, 1, \ldots)$

Ingredients

- **Compact** spaces $\mathcal{P}_n$ of n-supertiles $(n = 0, 1, \ldots)$
- For each $P_n \in \mathcal{P}_n$, geometric prescription for building P_n out of finitely many elements of $\mathcal{P}_{n-1}$.

Ingredients

- **Compact** spaces $\mathcal{P}_n$ of n-supertiles ($n = 0, 1, \ldots$)
- For each $P_n \in \mathcal{P}_n$, geometric prescription for building P_n out of finitely many elements of $\mathcal{P}_{n-1}$.
- Some geometric constraints on shapes and sizes of supertiles (van Hove)

Ingredients

- **Compact** spaces $\mathcal{P}_n$ of n-supertiles ($n = 0, 1, \ldots$)
- For each $P_n \in \mathcal{P}_n$, geometric prescription for building P_n out of finitely many elements of $\mathcal{P}_{n-1}$.
- Some geometric constraints on shapes and sizes of supertiles (van Hove)
- Weak primitivity: For each open $U_n \subset \mathcal{P}_n$, $\exists N$ s.t. every element of $\mathcal{P}_N$ contains an n-supertile from U_n.

Ingredients

- **Compact** spaces $\mathcal{P}_n$ of n-supertiles ($n = 0, 1, \ldots$)
- For each $P_n \in \mathcal{P}_n$, geometric prescription for building P_n out of finitely many elements of $\mathcal{P}_{n-1}$.
- Some geometric constraints on shapes and sizes of supertiles (van Hove)
- Weak primitivity: For each open $U_n \subset \mathcal{P}_n$, $\exists N$ s.t. every element of $\mathcal{P}_N$ contains an n-supertile from U_n.
- No strong primitivity! Smaller U_n's need bigger N's.

Ingredients

- **Compact** spaces $\mathcal{P}_n$ of n-supertiles $(n = 0, 1, \ldots)$
- For each $P_n \in \mathcal{P}_n$, geometric prescription for building P_n out of finitely many elements of $\mathcal{P}_{n-1}$.
- Some geometric constraints on shapes and sizes of supertiles (van Hove)
- Weak primitivity: For each open $U_n \subset \mathcal{P}_n$, $\exists N$ s.t. every element of $\mathcal{P}_N$ contains an n-supertile from U_n.
- No strong primitivity! Smaller U_n's need bigger N's.
- Weak primitivity implies repetitivity and minimality.

Dyadic solenoid

- $\mathcal{P}_n \simeq \mathbb{N} \cup \{\infty\}$. All n-supertiles A_i^n are intervals of length 2^n.

Dyadic solenoid

- $\mathcal{P}_n \simeq \mathbb{N} \cup \{\infty\}$. All n-supertiles A_i^n are intervals of length 2^n.
- $A_i^n = A_1^{n-1} A_{i+1}^{n-1}, \qquad A_\infty^n = A_1^{n-1} A_\infty^{n-1}$.

Dyadic solenoid

- $\mathcal{P}_n \simeq \mathbb{N} \cup \{\infty\}$. All n-supertiles A_i^n are intervals of length 2^n.
- $A_i^n = A_1^{n-1} A_{i+1}^{n-1}, \qquad A_\infty^n = A_1^{n-1} A_\infty^{n-1}.$
  - $A_i^1 = A_1 A_{i+1}$,
  - $A_i^2 = A_1 A_2 A_1 A_{i+2}$,
  - $A_i^3 = A_1 A_2 A_1 A_3 A_1 A_2 A_1 A_{i+3}$,
  - $A_i^4 = A_1 A_2 A_1 A_3 A_1 A_2 A_1 A_4 A_1 A_2 A_1 A_3 A_1 A_2 A_1 A_{i+4}$,

Dyadic solenoid

- $\mathcal{P}_n \simeq \mathbb{N} \cup \{\infty\}$. All n-supertiles A_i^n are intervals of length 2^n.
- $A_i^n = A_1^{n-1} A_{i+1}^{n-1}$, $\qquad A_\infty^n = A_1^{n-1} A_\infty^{n-1}$.
  - $A_i^1 = A_1 A_{i+1}$,
  - $A_i^2 = A_1 A_2 A_1 A_{i+2}$,
  - $A_i^3 = A_1 A_2 A_1 A_3 A_1 A_2 A_1 A_{i+3}$,
  - $A_i^4 = A_1 A_2 A_1 A_3 A_1 A_2 A_1 A_4 A_1 A_2 A_1 A_3 A_1 A_2 A_1 A_{i+4}$,
- Rule is the same at all levels. Can view as ILC *substitution*.

Dyadic solenoid

- $\mathcal{P}_n \simeq \mathbb{N} \cup \{\infty\}$. All n-supertiles A_i^n are intervals of length 2^n.
- $A_i^n = A_1^{n-1} A_{i+1}^{n-1}, \qquad A_\infty^n = A_1^{n-1} A_\infty^{n-1}$.
  - $A_i^1 = A_1 A_{i+1}$,
  - $A_i^2 = A_1 A_2 A_1 A_{i+2}$,
  - $A_i^3 = A_1 A_2 A_1 A_3 A_1 A_2 A_1 A_{i+3}$,
  - $A_i^4 = A_1 A_2 A_1 A_3 A_1 A_2 A_1 A_4 A_1 A_2 A_1 A_3 A_1 A_2 A_1 A_{i+4}$,
- Rule is the same at all levels. Can view as ILC *substitution*.
- Primitive, since A_i^n is contained in all $n+i$-supertiles, and any nbhd of A_∞^n contains some A_i^n.

Admissibility

- A patch P is *literally admissible* if it is found in a supertile.

Admissibility

- A patch P is *literally admissible* if it is found in a supertile.
- P is *admissible in the limit* if it is arbitrarily well approximated by literally admissible patches.

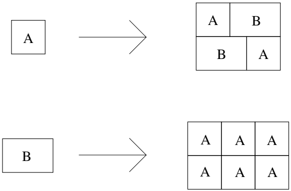
Admissibility

- A patch P is *literally admissible* if it is found in a supertile.
- P is *admissible in the limit* if it is arbitrarily well approximated by literally admissible patches.
- In dyadic solenoid, $A_1 A_\infty = A_\infty^1$ is literally admissible. $A_\infty A_1$ is admissible in the limit.

Admissibility

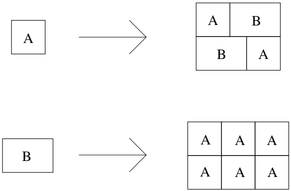
- A patch P is *literally admissible* if it is found in a supertile.
- P is *admissible in the limit* if it is arbitrarily well approximated by literally admissible patches.
- In dyadic solenoid, $A_1 A_\infty = A_\infty^1$ is literally admissible. $A_\infty A_1$ is admissible in the limit.
- Thm: In well-behaved fusions, tilings containing patches that are admissible in the limit have measure zero.

Shear between supertiles



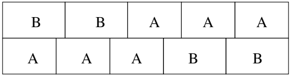
- Near top of n-supertile, essentially have $\sigma(a) = ab$, $\sigma(b) = aaa$.
- Near bottom boundary of n-supertile, $\sigma(a) = ba$, $\sigma(b) = aaa$.

Shear between supertiles



- Near top of n-supertile, essentially have $\sigma(a) = ab$, $\sigma(b) = aaa$.
- Near bottom boundary of n-supertile, $\sigma(a) = ba$, $\sigma(b) = aaa$.
- Non-Pisot. Discrepancies grow as $\lambda_2^n \to \infty$.

Shear in the limit



- Offsets are multiples of $|b|$ (mod $|a|$).
- Discrepancies grow without bound and $|b|/|a|$ irrational.
- Continuum of offsets appear *in the limit*.

Transition operators

- $P \in \mathcal{P}_n$, $Q \in \mathcal{P}_N$, $M_{n,N}(P,Q) =$ number of $P's$ in Q.

Transition operators

- $P \in \mathcal{P}_n$, $Q \in \mathcal{P}_N$, $M_{n,N}(P,Q) =$ number of $P's$ in Q.
- Column sum $\displaystyle\sum_P M_{n,N}(P,Q) = \#(n\text{-supertiles in } Q) < \infty$.
- If $n < m < N$, $M_{n,N}(P,Q) = \displaystyle\sum_{S \in \mathcal{P}_m} M_{n,m}(P,S) M_{m,N}(S,Q)$.

Three interpretations of $M_{n,N}$

An $n \times m$ matrix is an

- Array of nm numbers.

Three interpretations of $M_{n,N}$

An $n \times m$ matrix is an

- Array of nm numbers.
- Collection of m (column) vectors in $\mathbb{R}^n$.

Three interpretations of $M_{n,N}$

An $n \times m$ matrix is an

- Array of nm numbers.
- Collection of m (column) vectors in $\mathbb{R}^n$.
- Linear transformation $\mathbb{R}^m \to \mathbb{R}^n$.

Three interpretations of $M_{n,N}$

An $n \times m$ matrix is an

- Array of nm numbers.
- Collection of m (column) vectors in $\mathbb{R}^n$.
- Linear transformation $\mathbb{R}^m \to \mathbb{R}^n$.

Analogously,

- $M_{n,N}(P, Q)$ is a number ($\#(P$'s in $Q)$).

Three interpretations of $M_{n,N}$

An $n \times m$ matrix is an

- Array of nm numbers.
- Collection of m (column) vectors in $\mathbb{R}^n$.
- Linear transformation $\mathbb{R}^m \to \mathbb{R}^n$.

Analogously,

- $M_{n,N}(P,Q)$ is a number ($\#(P$'s in $Q)$).
- $M_{n,N}(*,Q)$ is a measure on $\mathcal{P}_n$.

Three interpretations of $M_{n,N}$

An $n \times m$ matrix is an

- Array of nm numbers.
- Collection of m (column) vectors in $\mathbb{R}^n$.
- Linear transformation $\mathbb{R}^m \to \mathbb{R}^n$.

Analogously,

- $M_{n,N}(P, Q)$ is a number ($\#(P$'s in $Q)$).
- $M_{n,N}(*, Q)$ is a measure on $\mathcal{P}_n$.
- $M_{n,N}$ is a map: (measures on $\mathcal{P}_N$) $\to$ (measures on $\mathcal{P}_n$).

Table of Contents

1. What does ILC look like?
2. Topological considerations
3. ILC fusions
4. Invariant Measures
5. Complexity
6. Tiling with infinitely many sizes
7. Summary

Measures on FLC fusion tiling spaces

- Need positive measure ρ_n on $\mathcal{P}_n$.
- Volume normalized: $\displaystyle\sum_{P \in \mathcal{P}_n} Vol(P)\rho_n(P) = 1.$

Measures on FLC fusion tiling spaces

- Need positive measure ρ_n on $\mathcal{P}_n$.
- Volume normalized: $\displaystyle\sum_{P \in \mathcal{P}_n} Vol(P)\rho_n(P) = 1$.
- $\rho_n = M_{n,N}\rho_N =$ linear combination of columns of $M_{n,N}$.

Measures on FLC fusion tiling spaces

- Need positive measure ρ_n on $\mathcal{P}_n$.
- Volume normalized: $\displaystyle\sum_{P \in \mathcal{P}_n} Vol(P)\rho_n(P) = 1$.
- $\rho_n = M_{n,N}\rho_N =$ linear combination of columns of $M_{n,N}$.
- $\Delta_{n,N} =$ (projectivized) cone spanned by columns of $M_{n,N}$.
- $\Delta_{n,\infty} =$ possible invariant measures on $\mathcal{P}_n$.

Measures on FLC fusion tiling spaces

- Need positive measure ρ_n on $\mathcal{P}_n$.
- Volume normalized: $\displaystyle\sum_{P \in \mathcal{P}_n} Vol(P)\rho_n(P) = 1$.
- $\rho_n = M_{n,N}\rho_N =$ linear combination of columns of $M_{n,N}$.
- $\Delta_{n,N} =$ (projectivized) cone spanned by columns of $M_{n,N}$.
- $\Delta_{n,\infty} =$ possible invariant measures on $\mathcal{P}_n$.
- $\displaystyle freq(P) = \lim_{n\to\infty} \sum_{Q \in \mathcal{P}_n} \rho_n(Q)(\#P \in Q)$

Adjustments to ILC spaces

- Still need positive measure ρ_n on $\mathcal{P}_n$.
- Still volume normalized: $\displaystyle\int_{\mathcal{P}_n} Vol(P)\rho_n(dP) = 1.$
- Need to make sense of:

Adjustments to ILC spaces

- Still need positive measure ρ_n on $\mathcal{P}_n$.
- Still volume normalized: $\displaystyle\int_{\mathcal{P}_n} Vol(P)\rho_n(dP) = 1$.
- Need to make sense of:
    - "Columns" of $M_{n,N}$.

Adjustments to ILC spaces

- Still need positive measure ρ_n on $\mathcal{P}_n$.
- Still volume normalized: $\displaystyle\int_{\mathcal{P}_n} Vol(P)\rho_n(dP) = 1$.
- Need to make sense of:
  - "Columns" of $M_{n,N}$.
  - Equation $\rho_n = M_{n,N}\rho_N$.

Adjustments to ILC spaces

- Still need positive measure ρ_n on $\mathcal{P}_n$.
- Still volume normalized: $\displaystyle\int_{\mathcal{P}_n} Vol(P)\rho_n(dP) = 1.$
- Need to make sense of:
    - "Columns" of $M_{n,N}$.
    - Equation $\rho_n = M_{n,N}\rho_N$.
    - Choquet simplices $\Delta_{n,N}$ and $\Delta_{n,\infty}$.

Adjustments to ILC spaces

- Still need positive measure ρ_n on $\mathcal{P}_n$.
- Still volume normalized: $\displaystyle\int_{\mathcal{P}_n} Vol(P)\rho_n(dP) = 1$.
- Need to make sense of:
    - "Columns" of $M_{n,N}$.
    - Equation $\rho_n = M_{n,N}\rho_N$.
    - Choquet simplices $\Delta_{n,N}$ and $\Delta_{n,\infty}$.
    - Frequencies of measureable sets of patches.

Columns of $M_{n,M}$

- Fix $Q \in \mathcal{P}_N$ and measurable $I \in \mathcal{P}_n$.
- $\zeta_{n,Q}(I) = \#(I \text{ in } Q) < \infty$.

Columns of $M_{n,M}$

- Fix $Q \in \mathcal{P}_N$ and measurable $I \in \mathcal{P}_n$.
- $\zeta_{n,Q}(I) = \#(I \text{ in } Q) < \infty$.
- Not volume normalized: $\displaystyle\int_{\mathcal{P}_n} Vol(P)\zeta_{n,Q}(dP) = Vol(Q)$.

$M_{n,N}$ as a map of measures

- Let ν be a measure on $\mathcal{P}_N$.
- Define $\mu := M_{n,N}(\nu)$ by $\mu(I) = \int_{\mathcal{P}_N} \zeta_{n,Q}(I)\nu(dQ)$
- μ is linear combination of $\zeta_{n,Q}$'s with weights given by ν.
- If $n < m < N$, $M_{n,N}(\nu) = M_{n,m}(M_{m,N}(\nu))$.
- $\int_{\mathcal{P}_n} Vol(P)\mu(dP) = \int_{\mathcal{P}_n \times \mathcal{P}_N} Vol(P)\zeta_{n,Q}(dP)\nu(dQ) = \int_{\mathcal{P}_N} Vol(Q)\nu(dQ)$.
- If ν is volume normalized, then so is μ.

$\Delta_{n,N}$ and $\Delta_{n,\infty}$

- $\Delta_{n,N}$ is projectivization of range of $M_{n,N}$. Can restrict to volume-normalized measures.

$\Delta_{n,N}$ and $\Delta_{n,\infty}$

- $\Delta_{n,N}$ is projectivization of range of $M_{n,N}$. Can restrict to volume-normalized measures.
- $M_{n,N+1} = M_{n,N} \circ M_{N,N+1}$, so $\Delta_{n,N+1} \subset \Delta_{n,N}$.
- $\Delta_{n,\infty} = \bigcap_{N>n} \Delta_{n,N}$.

$\Delta_{n,N}$ and $\Delta_{n,\infty}$

- $\Delta_{n,N}$ is projectivization of range of $M_{n,N}$. Can restrict to volume-normalized measures.
- $M_{n,N+1} = M_{n,N} \circ M_{N,N+1}$, so $\Delta_{n,N+1} \subset \Delta_{n,N}$.
- $\Delta_{n,\infty} = \bigcap_{N>n} \Delta_{n,N}$.
- Unique ergodicity means $\forall n, \Delta_{n,\infty} = \{\text{point}\}$.

Example: Dyadic Solenoid

- For every $Q \in \mathcal{P}_N$, $M_{0,N}(A_1, Q) = 2^{N-1} = \dfrac{1}{2} Vol(Q)$.

Example: Dyadic Solenoid

- For every $Q \in \mathcal{P}_N$, $M_{0,N}(A_1, Q) = 2^{N-1} = \frac{1}{2} Vol(Q)$.
- $\forall Q \in \mathcal{P}_N, \zeta_{0,Q}(A_1) = 2^{N-1}$.

Example: Dyadic Solenoid

- For every $Q \in \mathcal{P}_N$, $M_{0,N}(A_1, Q) = 2^{N-1} = \frac{1}{2} Vol(Q)$.
- $\forall Q \in \mathcal{P}_N, \zeta_{0,Q}(A_1) = 2^{N-1}$.
- Every measure in $\Delta_{0,N}$ gives frequency $1/2$ to A_1.
  Every measure in $\Delta_{0,N}$ gives frequency $(1/2)^k$ to A_k for all $k \leq N$.

Example: Dyadic Solenoid

- For every $Q \in \mathcal{P}_N$, $M_{0,N}(A_1, Q) = 2^{N-1} = \frac{1}{2} Vol(Q)$.
- $\forall Q \in \mathcal{P}_N, \zeta_{0,Q}(A_1) = 2^{N-1}$.
- Every measure in $\Delta_{0,N}$ gives frequency $1/2$ to A_1.
  Every measure in $\Delta_{0,N}$ gives frequency $(1/2)^k$ to A_k for all $k \leq N$.
- Every measure in $\Delta_{0,\infty}$ gives frequency $(1/2)^k$ to A_k and frequency 0 to A_∞.
  Every measure in $\Delta_{n,\infty}$ gives frequency $(1/2)^{n+k}$ to A^n_k and frequency 0 to A^n_∞.

Table of Contents

1 What does ILC look like?

2 Topological considerations

3 ILC fusions

4 Invariant Measures

5 Complexity

6 Tiling with infinitely many sizes

7 Summary

Sequences

For 1-sided sequences on a finite alphabet, usually define:

- $c(n) = \#($ words of length $n)$.
- If $c(n)$ is bounded, all sequences are eventually periodic.
- If $c(n) = n + 1$, sequence is either eventually periodic or Sturmian.
- For non-periodic substitutions, $k_1 n \leq c(n) \leq k_2 n$.
- Topological entropy is $\limsup \dfrac{\ln(c(n))}{n}$.

(d, ϵ)-separated sets

- Let X be a metric space with metric d. A (d, ϵ)-separated set is a set of points, no two of which are within distance ϵ.
- For 1-sided sequence space, define $d(T_1, T_2) =$(first location where $T_1 \neq T_2)^{-1}$.
- $c(n)$ is the maximum cardinality of a $1/n$-separated set.
- This depends on choice of metric on sequence space. Want something more robust.

The d_L metric

- Recall tiling distance: $d(T, T') = \inf_{\epsilon} |$ T and T' agree on $B_{1/\epsilon}$ up to rigid translation of up to ϵ.

The d_L metric

- Recall tiling distance: $d(T, T') = \inf_{\epsilon} |\; T$ and T' agree on $B_{1/\epsilon}$ up to rigid translation of up to ϵ.
- $d_L(T, T') = \sup_{x \in [0,L]^n} d(T - x, T' - x)$. Two tilings are ϵ-close if they agree on $[-\epsilon^{-1}, L + \epsilon^{-1}]^n$ up to ϵ changes in each tile.

The d_L metric

- Recall tiling distance: $d(T, T') = \inf_{\epsilon} |\ T$ and T' agree on $B_{1/\epsilon}$ up to rigid translation of up to ϵ.
- $d_L(T, T') = \sup_{x \in [0,L]^n} d(T - x, T' - x)$. Two tilings are ϵ-close if they agree on $[-\epsilon^{-1}, L + \epsilon^{-1}]^n$ up to ϵ changes in each tile.
- Let $C(\epsilon, L)$ be the maximal size of a (d_L, ϵ)-separated set.

The d_L metric

- Recall tiling distance: $d(T, T') = \inf_{\epsilon} |$ T and T' agree on $B_{1/\epsilon}$ up to rigid translation of up to ϵ.
- $d_L(T, T') = \sup_{x \in [0,L]^n} d(T - x, T' - x)$. Two tilings are ϵ-close if they agree on $[-\epsilon^{-1}, L + \epsilon^{-1}]^n$ up to ϵ changes in each tile.
- Let $C(\epsilon, L)$ be the maximal size of a (d_L, ϵ)-separated set.
- For suspensions of 1D subshifts, $C(\epsilon, L) \approx \epsilon^{-1} c(L + 2\epsilon^{-1})$.

The d_L metric

- Recall tiling distance: $d(T, T') = \inf_{\epsilon} |\ T$ and T' agree on $B_{1/\epsilon}$ up to rigid translation of up to ϵ.
- $d_L(T, T') = \sup_{x \in [0,L]^n} d(T - x, T' - x)$. Two tilings are ϵ-close if they agree on $[-\epsilon^{-1}, L + \epsilon^{-1}]^n$ up to ϵ changes in each tile.
- Let $C(\epsilon, L)$ be the maximal size of a (d_L, ϵ)-separated set.
- For suspensions of 1D subshifts, $C(\epsilon, L) \approx \epsilon^{-1} c(L + 2\epsilon^{-1})$.
- Scaling with L does not depend on ϵ or precise definition of d.

ILC tiling spaces

- Previous definitions do not require FLC.

ILC tiling spaces

- Previous definitions do not require FLC.
- $C(\epsilon, L)$ counts possible patches of size L, up to precision ϵ.
- Interesting questions involve fixing ϵ and taking $L \to \infty$.
  - If C bounded?
  - Is C polynomially bounded? ($C < f(\epsilon)(1+L)^\gamma$)
  - Entropy $= \limsup_{L \to \infty} \dfrac{\ln(C(\epsilon, L))}{L^n}$. Has units of $1/$Volume.

Invariance under conjugacy and metric change

Thm: Let Ω and Ω' be topologically conjugate tiling spaces with metrics d and d' and tiling complexity functions C and C'. Then for every $\epsilon > 0$ there exists an $\epsilon' > 0$ such that, for every L, $C(\epsilon, L) \leq C'(\epsilon', L)$.

Invariance under conjugacy and metric change

Thm: Let Ω and Ω' be topologically conjugate tiling spaces with metrics d and d' and tiling complexity functions C and C'. Then for every $\epsilon > 0$ there exists an $\epsilon' > 0$ such that, for every L, $C(\epsilon, L) \leq C'(\epsilon', L)$.

- If Ω' has bounded complexity, so does Ω.
- If Ω' has complexity that goes as L^γ, so does Ω.
- Entropy of Ω = entropy of Ω.

Invariance under homeomorphism

Thm: [Julien] Similar results apply when Ω and Ω' are merely homeomorphic, up to rescaling L by a fixed factor. Specifically,

Invariance under homeomorphism

Thm: [Julien] Similar results apply when Ω and Ω' are merely homeomorphic, up to rescaling L by a fixed factor. Specifically,

- If Ω' has bounded complexity, so does Ω.
- If Ω' has complexity that goes as L^γ, so does Ω.
- If Ω' has finite or zero entropy, so does Ω.

Example 1: Pinwheel

To define a patch of size L in the pinwheel tiling, you need to

- Say what sorts of supertiles are involved (finitely many choices).

Example 1: Pinwheel

To define a patch of size L in the pinwheel tiling, you need to

- Say what sorts of supertiles are involved (finitely many choices).
- Fix the rotations to within ϵ/L, since rotating by θ moves some points by $L\theta$. $O(L)$ choices.

Example 1: Pinwheel

To define a patch of size L in the pinwheel tiling, you need to

- Say what sorts of supertiles are involved (finitely many choices).
- Fix the rotations to within ϵ/L, since rotating by θ moves some points by $L\theta$. $O(L)$ choices.
- Fix the location of the origin to within ϵ. $O(L^2)$ choices.

Example 1: Pinwheel

To define a patch of size L in the pinwheel tiling, you need to

- Say what sorts of supertiles are involved (finitely many choices).
- Fix the rotations to within ϵ/L, since rotating by θ moves some points by $L\theta$. $O(L)$ choices.
- Fix the location of the origin to within ϵ. $O(L^2)$ choices.
- Total complexity is $O(L^3)$.

Self-similar tiling with shear but no rotations

If the origin is not within L of the boundary of high-order supertiles, only have $O(L^2)$ possibilities:

- Pick type of supertile that the origin is in. (Bounded choices)
- Specify the location of the origin ($O(L^2)$ possibilities).

Self-similar tiling with shear but no rotations

If the origin is not within L of the boundary of high-order supertiles, only have $O(L^2)$ possibilities:

- Pick type of supertile that the origin is in. (Bounded choices)
- Specify the location of the origin ($O(L^2)$ possibilities).

You need more data when the origin is near a fault line:

- Pick type of supertile that the origin is in. (Bounded choices)
- Specify the location of the origin in that supertile ($O(L^2)$ possibilities).

Self-similar tiling with shear but no rotations

If the origin is not within L of the boundary of high-order supertiles, only have $O(L^2)$ possibilities:

- Pick type of supertile that the origin is in. (Bounded choices)
- Specify the location of the origin ($O(L^2)$ possibilities).

You need more data when the origin is near a fault line:

- Pick type of supertile that the origin is in. (Bounded choices)
- Specify the location of the origin in that supertile ($O(L^2)$ possibilities).
- Specify the offset of the supertile across the fault line ($O(L)$ possibilities).

DPVs with FLC

Even if basic tiles are unit squares meeting full-edge-to-full-edge,

- Number of ways that 2 supertiles of size $\sim L$ can meet is $O(L)$.

DPVs with FLC

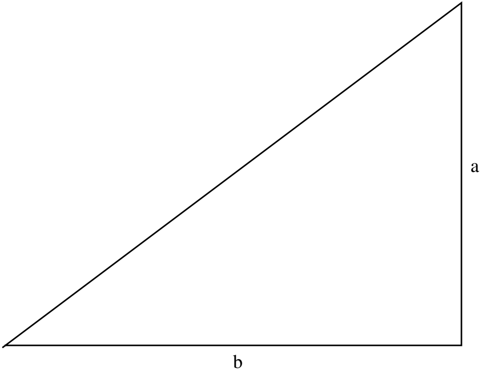
Even if basic tiles are unit squares meeting full-edge-to-full-edge,

- Number of ways that 2 supertiles of size $\sim L$ can meet is $O(L)$.
- Complexity goes as L^3 for self-similar. (Slightly different for self-affine).

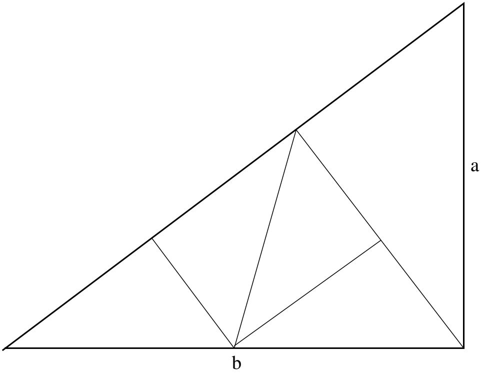
Table of Contents

1. What does ILC look like?
2. Topological considerations
3. ILC fusions
4. Invariant Measures
5. Complexity
6. Tiling with infinitely many sizes
7. Summary

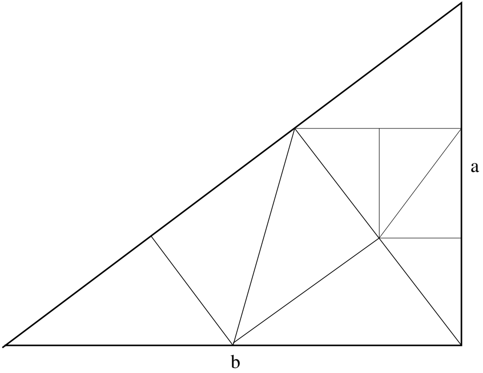
Generalized pinwheel



Generalized pinwheel



Generalized pinwheel



Properties of generalized pinwheel

- For all but countably many a/b, infinitely many directions.

Properties of generalized pinwheel

- For all but countably many a/b, infinitely many directions.
- For all but countably many a/b, infinitely many sizes.

Properties of generalized pinwheel

- For all but countably many a/b, infinitely many directions.
- For all but countably many a/b, infinitely many sizes.
- For all but countably many a/b, continuous shears.

Properties of generalized pinwheel

- For all but countably many a/b, infinitely many directions.
- For all but countably many a/b, infinitely many sizes.
- For all but countably many a/b, continuous shears.
- That's too complicated for today!
- To see how sizes work, let's do a 1D example instead.

Tiles and supertiles

- Tiling is 1D. Tiles are intervals.
- $\mathcal{P}_0 = [1, 3]$. Length of tile $P_0(x)$ is label x.

Tiles and supertiles

- Tiling is 1D. Tiles are intervals.
- $\mathcal{P}_0 = [1, 3]$. Length of tile $P_0(x)$ is label x.
- $\mathcal{P}_1 = [3/2, 3] \cup [3, (3/2)]$. In first interval, $P_1(x) = P_0(x)$: single tile of length x. In 2nd interval, $P_1(x) = P_0(x/3)P_0(2x/3)$.

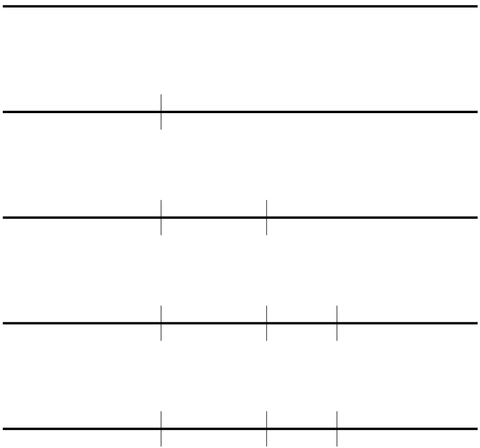
Tiles and supertiles

- Tiling is 1D. Tiles are intervals.
- $\mathcal{P}_0 = [1, 3]$. Length of tile $P_0(x)$ is label x.
- $\mathcal{P}_1 = [3/2, 3] \cup [3, (3/2)]$. In first interval, $P_1(x) = P_0(x)$: single tile of length x. In 2nd interval, $P_1(x) = P_0(x/3)P_0(2x/3)$.
- $\mathcal{P}_n = [(3/2)^n, 3(3/2)^n]$ with "special" points $3^a(3/2)^b$ doubled. If $x < 3(3/2)^{n-1}$, $P_n(x) = P_{n-1}(x)$; if $x > 3(3/2)^{n-1}$, $P_n(x) = P_{n-1}(x/3)P_{n-1}(2x/3)$.

Tiles and supertiles

- Tiling is 1D. Tiles are intervals.
- $\mathcal{P}_0 = [1,3]$. Length of tile $P_0(x)$ is label x.
- $\mathcal{P}_1 = [3/2, 3] \cup [3, (3/2)]$. In first interval, $P_1(x) = P_0(x)$: single tile of length x. In 2nd interval, $P_1(x) = P_0(x/3)P_0(2x/3)$.
- $\mathcal{P}_n = [(3/2)^n, 3(3/2)^n]$ with "special" points $3^a(3/2)^b$ doubled. If $x < 3(3/2)^{n-1}$, $P_n(x) = P_{n-1}(x)$; if $x > 3(3/2)^{n-1}$, $P_n(x) = P_{n-1}(x/3)P_{n-1}(2x/3)$.
- $\log(3)/\log(3/2)$ is irrational.

In pictures



Invariant measure

Transition matrix is sufficiently contracting that $\Delta_{n,\infty}$ is a single point. The invariant measure on $\mathcal{P}_n$ is $f_n(x)dx$, where

$$f_n(x) = \begin{cases} \dfrac{c}{x^2} & x < 2(3/2)^n \\ \dfrac{3c}{x^2} & x > 2(3/2)^n \end{cases}$$

and $c = 1/\ln(27/4)$. The special points have probability 0.

Table of Contents

1. What does ILC look like?
2. Topological considerations
3. ILC fusions
4. Invariant Measures
5. Complexity
6. Tiling with infinitely many sizes
7. Summary

Hierarchy is everywhere

- Atoms, cells, people, planets, galaxies, ...

Hierarchy is everywhere

- Atoms, cells, people, planets, galaxies, ...
- Strange attractors

Hierarchy is everywhere

- Atoms, cells, people, planets, galaxies, . . .
- Strange attractors
- Fractals

Hierarchy is everywhere

- Atoms, cells, people, planets, galaxies, . . .
- Strange attractors
- Fractals
- And of course tilings!

FLC substitution tilings are wonderful

- Substitution matrix tells you all about measures.
- If substitution is primitive, apply Perron-Frobenius theory.

FLC substitution tilings are wonderful

- Substitution matrix tells you all about measures.
- If substitution is primitive, apply Perron-Frobenius theory.
- Nonperiodicity = Recognizability.

FLC substitution tilings are wonderful

- Substitution matrix tells you all about measures.
- If substitution is primitive, apply Perron-Frobenius theory.
- Nonperiodicity = Recognizability.
- All eigenvalues are continuous.

FLC substitution tilings are wonderful

- Substitution matrix tells you all about measures.
- If substitution is primitive, apply Perron-Frobenius theory.
- Nonperiodicity = Recognizability.
- All eigenvalues are continuous.
- Short return vectors: For checking if $\vec{\alpha}$ is an eigenvalue, just need to see if $\lim \exp(2\pi i \lambda^n \vec{\alpha} \cdot \vec{v}) = 1$ for a finite collection of $\vec{v}$'s.

FLC substitution tilings are wonderful

- Substitution matrix tells you all about measures.
- If substitution is primitive, apply Perron-Frobenius theory.
- Nonperiodicity = Recognizability.
- All eigenvalues are continuous.
- Short return vectors: For checking if $\vec{\alpha}$ is an eigenvalue, just need to see if $\lim \exp(2\pi i \lambda^n \vec{\alpha} \cdot \vec{v}) = 1$ for a finite collection of $\vec{v}$'s.
- Anderson-Putnam construction.

FLC substitution tilings are wonderful

- Substitution matrix tells you all about measures.
- If substitution is primitive, apply Perron-Frobenius theory.
- Nonperiodicity $=$ Recognizability.
- All eigenvalues are continuous.
- Short return vectors: For checking if $\vec{\alpha}$ is an eigenvalue, just need to see if $\lim \exp(2\pi i \lambda^n \vec{\alpha} \cdot \vec{v}) = 1$ for a finite collection of $\vec{v}$'s.
- Anderson-Putnam construction.
- Collaring.

Fusion is:

- A way of understanding substitutions from the bottom up.

Fusion is:

- A way of understanding substitutions from the bottom up.
- A way of understanding hierarchies that aren't the same at every level.

Fusion is:

- A way of understanding substitutions from the bottom up.
- A way of understanding hierarchies that aren't the same at every level.
- A way of handling hierarchies of ILC tilings

Fusion is:

- A way of understanding substitutions from the bottom up.
- A way of understanding hierarchies that aren't the same at every level.
- A way of handling hierarchies of ILC tilings
- So general that you need to apply conditions to get anything useful.

Some nice FLC fusions

- Substitutions with non-self-similar tile sizes. (E.g. substitution sequences)

Some nice FLC fusions

- Substitutions with non-self-similar tile sizes. (E.g. substitution sequences)
- DPVs

Some nice FLC fusions

- Substitutions with non-self-similar tile sizes. (E.g. substitution sequences)
- DPVs
- Combinatorial substitutions and S-adic substitutions

Some nice FLC fusions

- Substitutions with non-self-similar tile sizes. (E.g. substitution sequences)
- DPVs
- Combinatorial substitutions and S-adic substitutions
- Method for generating counterexamples:
  - Minimal but not uniquely ergodic 1D example
  - Scrambled Fibonacci. Measurably conjugate to Fibonacci but with no continuous eigenvalues.

FLC fusion considerations

- Usually need to assume something about primitivity

FLC fusion considerations

- Usually need to assume something about primitivity
- Recognizability does not follow from non-periodicity.

FLC fusion considerations

- Usually need to assume something about primitivity
- Recognizability does not follow from non-periodicity.
- Measures controlled by transition matrices. Primitivity by itself does not imply unique ergodicity.

FLC fusion considerations

- Usually need to assume something about primitivity
- Recognizability does not follow from non-periodicity.
- Measures controlled by transition matrices. Primitivity by itself does not imply unique ergodicity.
- Spectra controlled by return vectors of n-supertiles in same $(n+2)$-supertile.

FLC fusion considerations

- Usually need to assume something about primitivity
- Recognizability does not follow from non-periodicity.
- Measures controlled by transition matrices. Primitivity by itself does not imply unique ergodicity.
- Spectra controlled by return vectors of n-supertiles in same $(n+2)$-supertile.
- Inverse limit structures are more complicated than Anderson-Putnam.

Some nice ILC fusions

- Tilings with rotations (pinwheel)
- Tilings with shears (many DPVs)
- Tilings with non-expansive dynamics (solenoid)
- Tilings with infinitely many tile shapes/sizes (generalized pinwheel)

Key ideas

- Replace finite sets $\mathcal{P}_n$ with compact sets.

Key ideas

- Replace finite sets $\mathcal{P}_n$ with compact sets.
- Adjust topology so patches can approximate one another.

Key ideas

- Replace finite sets $\mathcal{P}_n$ with compact sets.
- Adjust topology so patches can approximate one another.
- Admissibility in the limit.
- Measures and repetitivity act on open sets of patches.

Key ideas

- Replace finite sets $\mathcal{P}_n$ with compact sets.
- Adjust topology so patches can approximate one another.
- Admissibility in the limit.
- Measures and repetitivity act on open sets of patches.
- Replace transition matrix $(M_{n,N})_{ij}$ with transition operator $M_{n,N}(P,Q)$.
- Transition operators still control invariant measures.

Key ideas

- Replace finite sets $\mathcal{P}_n$ with compact sets.
- Adjust topology so patches can approximate one another.
- Admissibility in the limit.
- Measures and repetitivity act on open sets of patches.
- Replace transition matrix $(M_{n,N})_{ij}$ with transition operator $M_{n,N}(P, Q)$.
- Transition operators still control invariant measures.
- Complexity via counting (d_L, ϵ)-separated sets.
- Growth rate of complexity is topological invariant.

Thanks

Merci pour votre patience!