Hierarchy and fusion 3: Fusions with ILC

Lorenzo Sadun

University of Texas

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1. What does ILC look like?
Outline

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2. Topological considerations
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3. ILC fusions
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4. Invariant Measures
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5. Complexity

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6. Tiling with infinitely many sizes
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7. Summary
What is FLC?

- Finitely many tile types (prototiles) \( \{ t_i \} \).
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- Finitely many ways to have two tiles meet.
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- Finitely many patches of size $r$, up to translation.
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- Finitely many tile types (prototiles) \( \{ t_i \} \).
- Finitely many ways to have two tiles meet.
- Finitely many patches of size \( r \), up to translation.
- Excludes many interesting patterns.
Rotational ILC

Rotational ILC

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Shear ILC

What does ILC look like?
Topological considerations
ILC fusions
Invariant Measures
Complexity
Tiling with infinitely many sizes
Summary

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Combinatorial ILC

- Dyadic solenoid. Tiles $A_1, \ldots, A_\infty$.
- All tiles have length 1.
Combinatorial ILC

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- All tiles have length 1.
- Put $A_1$ in every other other other slot:
  ...$A_1$ $A_1$ $A_1$ $A_1$ $A_1$ ...
- Lather, rinse, repeat infinitely many times.
- If a slot is still empty, put $A_\infty$ there.
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Start with point set associated with FLC tiling.
Thermal ILC

- Start with point set associated with FLC tiling.
- Move each point by a small random amount.
Thermal ILC

- Start with point set associated with FLC tiling.
- Move each point by a small random amount.
- Build tiling from new points (e.g. from Voronoi cells).
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Is FLC the same as compactness?

- Usual tiling metric: $T$ and $T'$ are $\epsilon$-close if $T$ and $T'$ agree on $B_{1/\epsilon}$, up to rigid $\epsilon$-translation.
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- In that metric topology, $T$ has FLC $\iff \Omega_T$ is compact.
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- Abandon compactness or use different topology?
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- Abandon compactness or use different topology?
- Use different topology!
In FLC world, \{\text{tile labels}\} is finite.
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Space of tile labels

- In FLC world, \{tile labels\} is finite.
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  - For dyadic solenoid, \{labels\} = 1-point compactification of \(\mathbb{N}\).
Space of tile labels

- In FLC world, \{tile labels\} is finite.
- In ILC world, \{tile labels\} is \textit{compact}.
  - For pinwheel, \{labels\} = \(O(2)\).
  - For dyadic solenoid, \{labels\} = 1-point compactification of \(\mathbb{N}\).
- Geometry must be compatible: If \(\text{label}_i \to \text{label}_\infty\), then
  \((\text{shape of } t_i) \to (\text{shape of } t_\infty)\). (Hausdorff metric)
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Topology of ILC tiling spaces

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- In this topology, shears and rotations are continuous.
- In this topology, $\Omega_T$ is always compact.
Minimality and Repetitivity

A dynamical system is *minimal* if every orbit is dense.
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- Def: $T$ is repetitive if, for each $U$, $\exists R$ s.t. every ball of radius $R$ contains at least one patch from $U$. 
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$\Omega_T$ is minimal iff $T$ is repetitive.
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- **Compact** spaces $\mathcal{P}_n$ of $n$-supertiles ($n = 0, 1, \ldots$)
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- Weak primitivity: For each open $U_n \subset P_n$, $\exists N$ s.t. every element of $P_N$ contains an $n$-supertile from $U_n$. 

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- No strong primitivity! Smaller $U_n$’s need bigger $N$’s.
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- Weak primitivity: For each open $U_n \subset \mathcal{P}_n$, $\exists N$ s.t. every element of $\mathcal{P}_N$ contains an $n$-supertile from $U_n$.
- No strong primitivity! Smaller $U_n$’s need bigger $N$’s.
- Weak primitivity implies repetitivity and minimality.
Dyadic solenoid

- \( \mathcal{P}_n \cong \mathbb{N} \cup \{\infty\} \). All \( n \)-supertiles \( A^n_i \) are intervals of length \( 2^n \).
Dyadic solenoid

- $\mathcal{P}_n \simeq \mathbb{N} \cup \{\infty\}$. All $n$-supertiles $A^n_i$ are intervals of length $2^n$.
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  - $A^1_i = A_1 A_{i+1}$,
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Dyadic solenoid

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  - $A_i^1 = A_1 A_{i+1}$,
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- Rule is the same at all levels. Can view as ILC substitution.
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Dyadic solenoid

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- \( A^n_i = A^{n-1}_1 A^{n-1}_{i+1} \), \( A^n = A^{n-1}_1 A^{n-1}_\infty \).
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- Rule is the same at all levels. Can view as ILC substitution.
- Primitive, since \( A^n_i \) is contained in all \( n + i \)-supertiles, and any nbhd of \( A^n_\infty \) contains some \( A^n_i \).
A patch $P$ is *literally admissible* if it is found in a supertile.
Admissibility

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- $P$ is *admissible in the limit* if it is arbitrarily well approximated by literally admissible patches.
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In dyadic solenoid, $A_1 A_\infty = A_\infty^1$ is literally admissible. $A_\infty A_1$ is admissible in the limit.
A patch $P$ is **literally admissible** if it is found in a supertile.

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In dyadic solenoid, $A_1 A_\infty = A_\infty^1$ is literally admissible. $A_\infty A_1$ is admissible in the limit.

Thm: In well-behaved fusions, tilings containing patches that are admissible in the limit have measure zero.
Shear between supertiles

- Near top of $n$-supertile, essentially have $\sigma(a) = ab$, $\sigma(b) = aaa$.
- Near bottom boundary of $n$-supertile, $\sigma(a) = ba$, $\sigma(b) = aaa$. 
Shear between supertiles

- Near top of $n$-supertile, essentially have $\sigma(a) = ab$, $\sigma(b) = aaa$.
- Near bottom boundary of $n$-supertile, $\sigma(a) = ba$, $\sigma(b) = aaa$.
- Non-Pisot. Discrepancies grow as $\lambda_2^n \to \infty$. 
Shear in the limit

- Offsets are multiples of $|b| \pmod{|a|}$.
- Discrepancies grow without bound and $|b|/|a|$ irrational.
- Continuum of offsets appear in the limit.
Transition operators

\[ P \in \mathcal{P}_n, \ Q \in \mathcal{P}_N, \ M_{n,N}(P, Q) = \text{number of } P\text{'s in } Q. \]
Transition operators

- \( P \in \mathcal{P}_n, \ Q \in \mathcal{P}_N, \ M_{n,N}(P, Q) = \text{number of } P' \text{s in } Q. \)
- Column sum \( \sum_P M_{n,N}(P, Q) = \#(n\text{-supertiles in } Q) < \infty. \)
- If \( n < m < N, \ M_{n,N}(P, Q) = \sum_{S \in \mathcal{P}_m} M_{n,m}(P, S)M_{m,N}(S, Q). \)
Three interpretations of $M_{n,N}$

An $n \times m$ matrix is an

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Three interpretations of $M_{n,N}$

An $n \times m$ matrix is an

- Array of $nm$ numbers.
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- Linear transformation $\mathbb{R}^m \rightarrow \mathbb{R}^n$. 
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Analogously,

- $M_{n,N}(P, Q)$ is a number ($\#(P's \ in \ Q)$).
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- $M_{n,N}(\ast, Q)$ is a measure on $\mathcal{P}_n$. 
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Analogously,

- $M_{n,N}(P, Q)$ is a number ($\#(P$'s in $Q$)).
- $M_{n,N}(*, Q)$ is a measure on $\mathcal{P}_n$.
- $M_{n,N}$ is a map: (measures on $\mathcal{P}_N$) $\rightarrow$ (measures on $\mathcal{P}_n$).
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Measures on FLC fusion tiling spaces

- Need positive measure \( \rho_n \) on \( \mathcal{P}_n \).
- Volume normalized: \[ \sum_{P \in \mathcal{P}_n} \text{Vol}(P) \rho_n(P) = 1. \]
Measures on FLC fusion tiling spaces

- Need positive measure $\rho_n$ on $\mathcal{P}_n$.
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- $\rho_n = M_{n,N}\rho_N = \text{linear combination of columns of } M_{n,N}$. 
Measures on FLC fusion tiling spaces

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- $\rho_n = M_{n,N} \rho_N$ = linear combination of columns of $M_{n,N}$.
- $\Delta_{n,N}$ = (projectivized) cone spanned by columns of $M_{n,N}$.
- $\Delta_{n,\infty}$ = possible invariant measures on $\mathcal{P}_n$. 
Measures on FLC fusion tiling spaces

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- $\Delta_{n,N} = \text{(projectivized) cone spanned by columns of } M_{n,N}$.
- $\Delta_{n,\infty} = \text{possible invariant measures on } \mathcal{P}_n$.
- $freq(P) = \lim_{n \to \infty} \sum_{Q \in \mathcal{P}_n} \rho_n(Q)(\#P \in Q)$
Adjustments to ILC spaces

- Still need positive measure $\rho_n$ on $\mathcal{P}_n$.
- Still volume normalized: $\int_{\mathcal{P}_n} Vol(P)\rho_n(dP) = 1$.
- Need to make sense of:
Adjustments to ILC spaces

- Still need positive measure $\rho_n$ on $\mathcal{P}_n$.
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  - “Columns” of $M_{n,N}$.
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  - Equation $\rho_n = M_{n,N} \rho_N$. 
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  - Choquet simplices $\Delta_{n,N}$ and $\Delta_{n,\infty}$.
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  - Equation $\rho_n = M_{n,N}\rho_N$.
  - Choquet simplices $\Delta_{n,N}$ and $\Delta_{n,\infty}$.
  - Frequencies of measureable sets of patches.
Columns of $M_{n,M}$

- Fix $Q \in \mathcal{P}_N$ and measurable $I \in \mathcal{P}_n$.
- $\zeta_{n,Q}(I) = \#(I \text{ in } Q) < \infty$. 

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Columns of $M_{n,M}$

- Fix $Q \in \mathcal{P}_N$ and measurable $I \in \mathcal{P}_n$.
- $\zeta_{n,Q}(I) = \#(I \text{ in } Q) < \infty$.
- Not volume normalized: $\int_{\mathcal{P}_n} \text{Vol}(P) \zeta_{n,Q}(dP) = \text{Vol}(Q)$. 

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Let $\nu$ be a measure on $P_N$.

Define $\mu := M_{n,N}(\nu)$ by $\mu(I) = \int_{P_N} \zeta_{n,Q}(I) \nu(dQ)$

$\mu$ is linear combination of $\zeta_{n,Q}$’s with weights given by $\nu$.

If $n < m < N$, $M_{n,N}(\nu) = M_{n,m}(M_{m,N}(\nu))$.

$$\int_{P_n} \text{Vol}(P)\mu(dP) = \int_{P_n \times P_N} \text{Vol}(P)\zeta_{n,Q}(dP)\nu(dQ) = \int_{P_N} \text{Vol}(Q)\nu(dQ).$$

If $\nu$ is volume normalized, then so is $\mu$.
\( \Delta_{n,N} \) and \( \Delta_{n,\infty} \)

- \( \Delta_{n,N} \) is projectivization of range of \( M_{n,N} \). Can restrict to volume-normalized measures.
**Δₙ,₇ and Δₙ,∞**

- Δₙ,₇ is projectivization of range of Mₙ,₇. Can restrict to volume-normalized measures.
- Mₙ,₇⁺₁ = Mₙ,₇ ∘ M₇,₇⁺₁, so Δₙ,₇⁺₁ ⊂ Δₙ,₇.
- Δₙ,∞ = ℂ ∩ Δₙ,₇.
$\Delta_{n,N}$ and $\Delta_{n,\infty}$

- $\Delta_{n,N}$ is projectivization of range of $M_{n,N}$. Can restrict to volume-normalized measures.
- $M_{n,N+1} = M_{n,N} \circ M_{N,N+1}$, so $\Delta_{n,N+1} \subset \Delta_{n,N}$.
- $\Delta_{n,\infty} = \bigcap_{N>n} \Delta_{n,N}$.
- Unique ergodicity means $\forall n, \Delta_{n,\infty} = \{\text{point}\}$. 

$\Delta_{n,N}$ and $\Delta_{n,\infty}$
Example: Dyadic Solenoid

For every $Q \in \mathcal{P}_N$, $M_{0,N}(A_1, Q) = 2^{N-1} = \frac{1}{2} \text{Vol}(Q)$. 

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Hierarchy and fusion 3: Fusions with ILC
Example: Dyadic Solenoid

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- $\forall Q \in \mathcal{P}_N, \zeta_{0,Q}(A_1) = 2^{N-1}$. 
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- Every measure in $\Delta_{0,\infty}$ gives frequency $(1/2)^k$ to $A_k$ and frequency 0 to $A_\infty$.
  Every measure in $\Delta_{n,\infty}$ gives frequency $(1/2)^{n+k}$ to $A_{k}^{n}$ and frequency 0 to $A_{\infty}^{n}$. 
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1. What does ILC look like?
2. Topological considerations
3. ILC fusions
4. Invariant Measures
5. Complexity
6. Tiling with infinitely many sizes
7. Summary
Sequences

For 1-sided sequences on a finite alphabet, usually define:

- $c(n) = \#(\text{words of length } n)$.
- If $c(n)$ is bounded, all sequences are eventually periodic.
- If $c(n) = n + 1$, sequence is either eventually periodic or Sturmian.
- For non-periodic substitutions, $k_1 n \leq c(n) \leq k_2 n$.
- Topological entropy is $\limsup \frac{\ln(c(n))}{n}$.
Let $X$ be a metric space with metric $d$. A $(d, \epsilon)$-separated set is a set of points, no two of which are within distance $\epsilon$.

For 1-sided sequence space, define
\[ d(T_1, T_2) = (\text{first location where } T_1 \neq T_2)^{-1}. \]

$c(n)$ is the maximum cardinality of a $1/n$-separated set.

This depends on choice of metric on sequence space. Want something more robust.
The $d_L$ metric

- Recall tiling distance: $d(T, T') = \inf_{\epsilon} | T$ and $T'$ agree on $B_{1/\epsilon}$ up to rigid translation of up to $\epsilon$. 
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- $d_L(T, T') = \sup_{x \in [0, L]^n} d(T - x, T' - x)$. Two tilings are $\varepsilon$-close if they agree on $[-\varepsilon^{-1}, L + \varepsilon^{-1}]^n$ up to $\varepsilon$ changes in each tile.
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- Let $C(\epsilon, L)$ be the maximal size of a $(d_L, \epsilon)$-separated set.
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- Let $C(\epsilon, L)$ be the maximal size of a $(d_L, \epsilon)$-separated set.
- For suspensions of 1D subshifts, $C(\epsilon, L) \approx \epsilon^{-1} c(L + 2\epsilon^{-1}).$
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- Let $C(\epsilon, L)$ be the maximal size of a $(d_L, \epsilon)$-separated set.
- For suspensions of 1D subshifts, $C(\epsilon, L) \approx \epsilon^{-1} c(L + 2\epsilon^{-1})$.
- Scaling with $L$ does not depend on $\epsilon$ or precise definition of $d$. 

Recall the $d_L$ metric, which measures the distance between two tilings $T$ and $T'$ as the supremum over all points $x$ in the interval $[0, L]^n$ of the $\ell_1$ distance between $T - x$ and $T' - x$. This distance is used to quantify how close two tilings are, taking into account their agreement up to rigid translations. The function $C(\epsilon, L)$ represents the maximal size of a set that is $(d_L, \epsilon)$-separated, meaning that any two points in this set are at least $\epsilon$ apart in the $d_L$ metric. The estimate $C(\epsilon, L) \approx \epsilon^{-1} c(L + 2\epsilon^{-1})$ shows how the maximal size scales with the size of the tiling $L$ and the separation parameter $\epsilon$. This scaling behavior is independent of $\epsilon$ and the precise definition of the metric $d$. 

The $d_L$ metric is a fundamental tool in the study of tiling spaces and their properties, particularly in the context of dynamical systems and symbolic dynamics.
ILC tiling spaces

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- $C(\epsilon, L)$ counts possible patches of size $L$, up to precision $\epsilon$.
- Interesting questions involve fixing $\epsilon$ and taking $L \to \infty$.
  - If $C$ bounded?
  - Is $C$ polynomially bounded? ($C < f(\epsilon)(1 + L)^\gamma$)
  - Entropy = $\limsup_{L \to \infty} \frac{\ln(C(\epsilon, L))}{L^n}$. Has units of $1/\text{Volume}$. 
Invariance under conjugacy and metric change

Thm: Let $\Omega$ and $\Omega'$ be topologically conjugate tiling spaces with metrics $d$ and $d'$ and tiling complexity functions $C$ and $C'$. Then for every $\epsilon > 0$ there exists an $\epsilon' > 0$ such that, for every $L$, $C(\epsilon, L) \leq C'(\epsilon', L)$. 

If $\Omega'$ has bounded complexity, so does $\Omega$. If $\Omega'$ has complexity that goes as $L^\gamma$, so does $\Omega$. Entropy of $\Omega = \text{entropy of } \Omega'$. 

Lorenzo Sadun
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Invariance under homeomorphism

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- If $\Omega'$ has bounded complexity, so does $\Omega$.
- If $\Omega'$ has complexity that goes as $L^\gamma$, so does $\Omega$.
- If $\Omega'$ has finite or zero entropy, so does $\Omega$. 
Example 1: Pinwheel

To define a patch of size $L$ in the pinwheel tiling, you need to:
- Say what sorts of supertiles are involved (finitely many choices).
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- Fix the location of the origin to within $\epsilon$. $O(L^2)$ choices.
- Total complexity is $O(L^3)$. 
If the origin is not within $L$ of the boundary of high-order supertiles, only have $O(L^2)$ possibilities:

- Pick type of supertile that the origin is in. (Bounded choices)
- Specify the location of the origin ($O(L^2)$ possibilities).
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- Specify the offset of the supertile across the fault line ($O(L)$ possibilities).
Even if basic tiles are unit squares meeting full-edge-to-full-edge,

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- Number of ways that 2 supertiles of size $\sim L$ can meet is $O(L)$.
- Complexity goes as $L^3$ for self-similar. (Slightly different for self-affine).
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- For all but countably many $a/b$, infinitely many sizes.
- For all but countably many $a/b$, continuous shears.
- That’s too complicated for today!
- To see how sizes work, let’s do a 1D example instead.
Tiles and supertiles

- Tiling is 1D. Tiles are intervals.
- $\mathcal{P}_0 = [1, 3]$. Length of tile $P_0(x)$ is label $x$. 

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- $\mathcal{P}_n = [(3/2)^n, 3(3/2)^n]$ with “special” points $3^a(3/2)^b$ doubled. If $x < 3(3/2)^{n-1}$, $P_n(x) = P_{n-1}(x)$; if $x > 3(3/2)^{n-1}$, $P_n(x) = P_{n-1}(x/3)P_{n-1}(2x/3)$. 
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- $\log(3)/\log(3/2)$ is irrational.
In pictures
Transition matrix is sufficiently contracting that $\Delta_{n,\infty}$ is a single point. The invariant measure on $\mathcal{P}_n$ is $f_n(x)dx$, where

$$f_n(x) = \begin{cases} \frac{c}{x^2} & x < 2(3/2)^n \\ \frac{3c}{x^2} & x > 2(3/2)^n \end{cases}$$

and $c = 1/\ln(27/4)$. The special points have probability 0.
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- And of course tilings!
FLC substitution tilings are wonderful

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Short return vectors: For checking if $\vec{\alpha}$ is an eigenvalue, just need to see if $\lim \exp(2\pi i \lambda^n \vec{\alpha} \cdot \vec{v}) = 1$ for a finite collection of $\vec{v}$'s.
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- Collaring.
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- A way of handling hierarchies of ILC tilings
- So general that you need to apply conditions to get anything useful.
Some nice FLC fusions

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- DPVs
- Combinatorial substitutions and S-adic substitutions
- Method for generating counterexamples:
  - Minimal but not uniquely ergodic 1D example
  - Scrambled Fibonacci. Measurably conjugate to Fibonacci but with no continuous eigenvalues.
FLC fusion considerations

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- Measures controlled by transition matrices. Primitivity by itself does not imply unique ergodicity.
- Spectra controlled by return vectors of $n$-supertiles in same $(n + 2)$-supertile.
- Inverse limit structures are more complicated than Anderson-Putnam.
Some nice ILC fusions

- Tilings with rotations (pinwheel)
- Tilings with shears (many DPVs)
- Tilings with non-expansive dynamics (solenoid)
- Tilings with infinitely many tile shapes/sizes (generalized pinwheel)
Key ideas

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- Adjust topology so patches can approximate one another.
- Admissibility in the limit.
- Measures and repetitivity act on open sets of patches.
- Replace transition matrix $(M_{n,N})_{ij}$ with transition operator $M_{n,N}(P, Q)$.
- Transition operators still control invariant measures.
- Complexity via counting $(d_L, \epsilon)$-separated sets.
- Growth rate of complexity is topological invariant.
Merci pour votre patience!