Hierarchy and fusion 2: Fusions with FLC

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September 19, 2017

Outline

1. Two views of substitution

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2. Combinatorial Substitutions

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- Finitely many tile types (prototiles) $\{t_i\}$.
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- "Substitution rule" for replacing $L(t_i)$ with collection $\sigma(t_i)$ of tiles: *1-supertile*.
- Extend σ to patches, tilings and tiling spaces.
- Tiling T is *admissible* if every patch is found in an n-supertile $\sigma^n(t_i)$. $\Omega_T = \{\text{admissible tilings}\}$.

An alternate approach

- Look for admissible T such that $\sigma(T) = T$. "Self-similar" tiling.
- (May need to replace σ by power of σ. σ_{TM} has no fixed point, but σ_{TM}^2 has four.)

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- I'm not one of them. Back to spaces!

What is an (n+1)-supertile?

- $\sigma^{n+1}(t_i) = \sigma(\sigma^n(t_i))$.
- To get an $(n+1)$-supertile I_{n+1} of type i, start with an n-supertile I_n of type i and apply the substitution to each tile of I_n.

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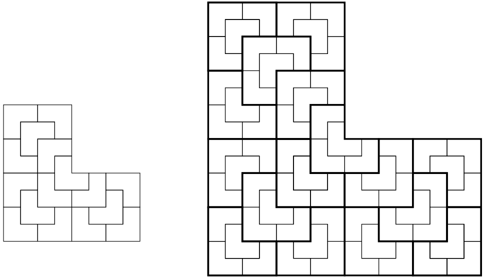

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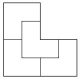

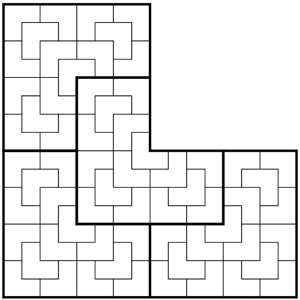

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A 1D example

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- $M = \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}$, eigenvalues $(1 \pm \sqrt{13})/2$.

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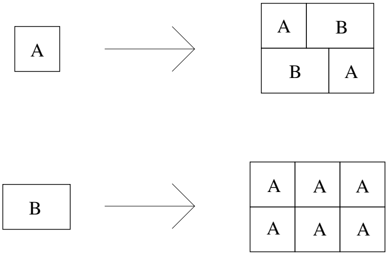
- For *combinatorial* substitution can take $|a|$, $|b|$ arbitrary. (E.g. $|a| = |b| = 1$).
- $\sigma(\text{word}) =$ concatenation of $\sigma(\text{each letter})$.
- History: Substitution *sequences* came long before substitution *tilings*. For Pisot substitutions, dynamics are the same for both. For other substitutions, they aren't.

A 2D example



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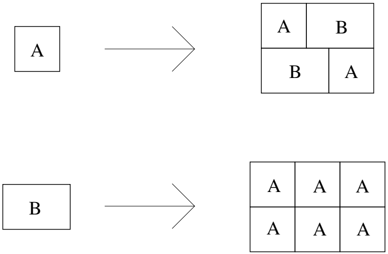

- Direct product *variation* (DPV)
- For geometric substitution, need (B width/A width)=$(\sqrt{13}-1)/2$, A height $=$ B height. $L = \begin{pmatrix} (\sqrt{13}+1)/2 & 0 \\ 0 & 2 \end{pmatrix}$. Tiling does not have FLC.
- For combinatorial substitution, can use unit squares.

Troubles with geometry

A

A	B
B	A

A	B	A	A	A
B	A	A	A	A
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Example is not really a substitution!

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- $\Omega_\sigma = \{$ all admissible tilings$\}$.

What stays the same?

- Fixed substitution matrix M. M_{ij} = number of n-supertiles of type i in each $(n+1)$ supertile of type j.
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- Perron-Frobenius theory.
- Can write Ω_σ as inverse limit.
- Can compute cohomology using variants of Anderson-Putnam.

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- Spectral theory requires geometric data.

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Revised rules of the game

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- For each $n > 0$, geometric rule for assembling each n-supertile from $(n-1)$-supertiles.
- Transition matrices $(M_{n,N})_{ij}$ saying how many I_n's are in J_N. $M_{n,N} = M_{n,n+1} M_{n+1,n+2} \cdots M_{N-1,N}$.
- (Some choices of) Supertiles form van Hove sequence.

Too much freedom

Every FLC tiling space is obtained from a fusion:

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- Hierarchy exists but is meaningless.
- To get *meaningful* results, must apply constraints.

Other desirable conditions

- Recognizability: For each n there is a radius D_n s.t. if two tilings agree on B_{r+D_n}, then their supertiles of level n agree on B_r.
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- Weak primitivity: For each n there is an N s.t. every N-supertile contains at least one of each n-supertile. (For each n, $\exists N$ s.t. $M_{n,N}$ is primitive.)
- Strong primitivity: For each n, every $(n+1)$-supertile contains at least one of each n-supertile. (For each n, $M_{n,n+1}$ is primitive.)

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- Matrix boundedness: $\sum_{ij} (M_{n,n+1})_{ij}$ is uniformly bounded.
- Transition-regularity: $M_{n,n+1}$ is the same for every n.

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- Sacrifices matrix boundedness.

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- $freq_\mu(P) := \mu([P])$.
- If μ is ergodic, μ-a.e. T has the property that

$$\lim_{r\to\infty} \frac{\#(P \in B_r)}{Vol(B_r)} = freq_\mu(P).$$

Well-Defined Supertile Frequencies

- Define $\rho_n(i)$ = frequency of n-supertile of type i.
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- $\Delta_{n,N+1} \subset \Delta_{n,N}$, since $M_{n,N+1} = M_{n,N} M_{N,N+1}$.
- Limiting simplex $\Delta_{n,\infty}$ describes all possible n-supertile frequencies.
- Ergodic measures are vertices of simplex.

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Unique ergodicity

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- Then ergodic theorem applies to *every* tiling, not just a.e.
- All tilings have the same naive patch frequencies.
- Warning: Unique ergodicity does *not* follow from primitivity or strong primitivity. Need control on transition matrices.

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- $M_{n-1,n} = \begin{pmatrix} 10^n & 1 \\ 1 & 10^n \end{pmatrix}$.
- A_N has about 90% a's and 10% b's
  99% A_1's and 1% B_1's
  99.9% A_2's and 0.1% B_2's, etc.
- B_N is the opposite.

Quantitative control over the Choquet complex

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- Claim: $Diam(\Delta_{n,N+1})/Diam(\Delta_{n,N}) \leq 1 - \delta(M_{N,N+1})$.
- Corollary: If $\sum_N \delta(M_{N,N+1})$ diverges, system is uniquely ergodic.
- In previous example, $\sum_N \delta(M_{N,N+1}) \sim \sum 10^{-N} < \infty$.

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Primitivity Assumptions

- Fusion is strongly primitive.
- If only weakly primitive, use acceleration to make it strongly primitive.

When is $\vec{\alpha}$ a topological eigenvalue?

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- I.e., iff, $\forall \epsilon > 0$, $\exists R$ s.t., whenever two patches of inner radius R are separated by displacement $\vec{x}$, $|\exp(2\pi i \vec{\alpha} \cdot \vec{x}) - 1| < \epsilon$.

When is $\vec{\alpha}$ a topological eigenvalue?

- $\vec{\alpha} \in \mathbb{R}^n$ is top. e-vec iff, $\exp(2\pi i \vec{\alpha} \cdot \vec{x})$ is continuous on $\{T - \vec{x}\}$.
- I.e., iff, $\forall \epsilon > 0$, $\exists R$ s.t., whenever two patches of inner radius R are separated by displacement $\vec{x}$, $|\exp(2\pi i \vec{\alpha} \cdot \vec{x}) - 1| < \epsilon$.
- To say more, need way to write arbitrary return vector as sum of standard pieces.

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- For ordinary substitutions, $\mathcal{V}^n = \lambda \mathcal{V}^{n-1}$, and either $\eta_n(\vec{\alpha}) \to 0$ exponentially or $\eta_n(\vec{\alpha}) \not\to 0$.
- For strongly primitive *substitution* tilings, $\vec{\alpha}$ is an eigenvalue iff, $\forall v \in \mathcal{V}^0$, $\lim_n \exp(2\pi i \lambda^n \vec{\alpha} \cdot \vec{v}) = 1$.

Sketch of proof

- Lemma: If $\vec{x}$ and $\vec{y}$ are corresponding points in n-supertiles in the same N-supertile, then $\vec{y} - \vec{x} = \displaystyle\sum_{k=n}^{N-2} \vec{v}_k$, for some $\vec{v}_k \in \mathcal{V}^k$.

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- If P is big, P contains an n-supertile, so $\exists N$ s.t. $\vec{y} - \vec{x} = \sum_{k=n}^{N-2} \vec{v}_k$.
- If P is big enough and $\sum \eta_k$ converges, $|\exp(2\pi i \vec{\alpha} \cdot y) - \exp(2\pi i \vec{\alpha} \cdot x)| \leq \sum_{k=n}^{\infty} \eta_k(\vec{\alpha}) < \epsilon$.

Sketch of the converse

- For converse, pick $v_k \in \mathcal{V}^k$ with $|\exp(2\pi i \vec{\alpha} \cdot \vec{v}_k) - 1| = \eta_k(\vec{\alpha})$. (Worse case scenario)

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- If P lies in an n-supertile, can arrange for for $\vec{y} - \vec{x} = v_n + v_{n+3} + v_{n+6} + \cdots + v_{n+3m}$ with m arbitrary.
- If $\sum \eta_k(\vec{\alpha})$ diverges, no matter how big n is, can arrange for $|\exp(2\pi i\vec{\alpha} \cdot y) - \exp(2\pi i\vec{\alpha} \cdot x)| > \epsilon$.

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- Scrambled Fibonacci tiling has pure point measurable spectrum, but *no* nonzero continuous eigenvalues.

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- 1D substitution on two letters. $\sigma(a) = ab$, $\sigma(b) = a$.
- Denote n-supertiles as $F_n(a)$ and $F_n(b)$.
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- $M_0 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $\lambda = \phi = (1+\sqrt{5})/2$.
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- Measurable spectrum is pure point: $\dfrac{1}{\sqrt{5}}\mathbb{Z}[\phi]$.

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  E.g. can take $N_n = 3^n$.
- Define $A_n(a) = F_{N_n}(a)$, $A_n(b) = F_{N_n}(b)$.
- $M_{n,n+1} = M_0^{N_{n+1}-N_n}$.

Scrambled Fibonacci

- Tweak accelerated Fibonacci by introducing 3rd supertile type $S_{2n+1}(c)$ at odd levels.
- $S_{2n+1}(a)$ and $S_{2n+1}(b)$ are built from $S_{2n}(a)$ and $S_{2n}(b)$ in same way that A_{2n+1}'s are built from A_{2n}'s.
- $S_{2n+1}(c)$ has same population as $S_{2n+1}(b)$, except all $S_{2n}(a)$'s come before any $S_{2n}(b)$'s. $S_{2n+1}(c)$ have *very* long periodic stretches.
- $S_{2n}(a)$ and $S_{2n}(b)$ are built from $S_{2n-1}(a)$ and $S_{2n-1}(b)$ in same way that A_{2n}'s are built from A_{2n-1}'s, except replacing *one* $S_{2n-1}(b)$ with $S_{2n-1}(c)$. $S_{2n-1}(c)$'s are *very* rare.

Killing off the topological eigenvalues

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- But $S_{2n+1}(c)$ has $\sim \phi^{N_{2n+1}-N_{2n}}$ consecutive $S_{2n}(a)$'s.
- Since $N_{2n+1} - N_{2n} \gg N_{2n}$, things get out of phase.

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- Run into trouble if origin is in an $S_{2n+1}(c)$, but can deal with finitely many exceptions.
- If $\sum_n \phi^{N_{2n-1}-N_{2n}}$ converges, then probability that the origin is in $E_{2n+1}(c)$ for infinitely many n's is zero. Map is defined almost everywhere.

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- No! The vertices of scrambled Fibonacci are repetitive FLC, thanks to primitivity.
- The vertices have pure point diffraction spectrum, thanks to topological conjugacy to Fibonacci.
- The vertices do not have the Meyer property, so cannot be a model set.