Hierarchy and fusion 1: Substitutions

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Outline

1. What is the world made of?
   - Types of matter
   - Hierarchy

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   - Types of matter
   - Hierarchy
2. Examples

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Fluids

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- Large-scale properties (like pressure) are described by laws of probability.

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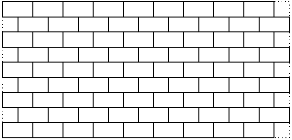

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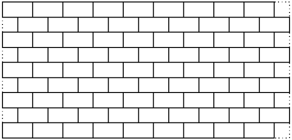

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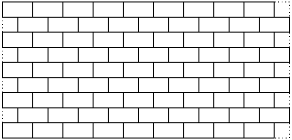

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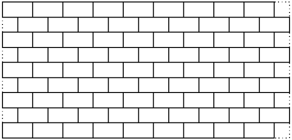

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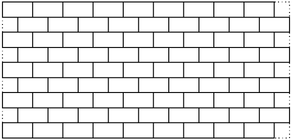

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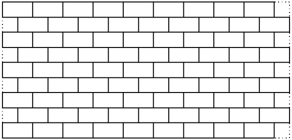

- Understand one part of the wall and you understand the rest.
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- We call this behavior "periodic" (aka "boring").

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- The arrangement of ingredients is random.
- If you understand crystals and fluids, you understand mixtures.

Everything else

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Everything else

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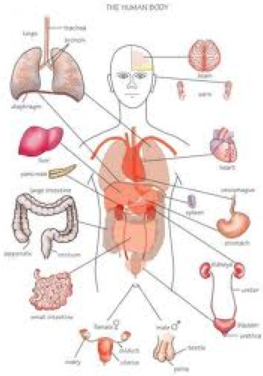
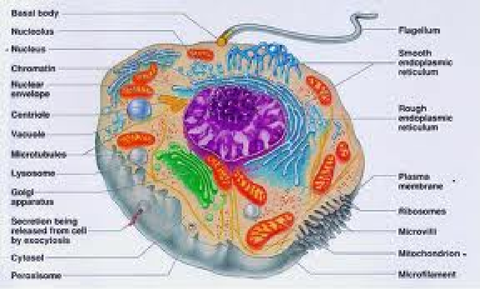
- Almost everything you see is made up of definite parts.
- Each part is made up of smaller parts.
- Smaller parts are made up of still smaller parts, etc.
- There is interesting structure at many different scales.
- An object with many levels of organization is called hierarchical

People are made of organs, tissues, and cells



Cells are made from parts



Cell parts are made from proteins

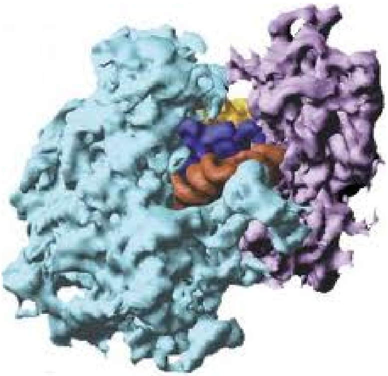
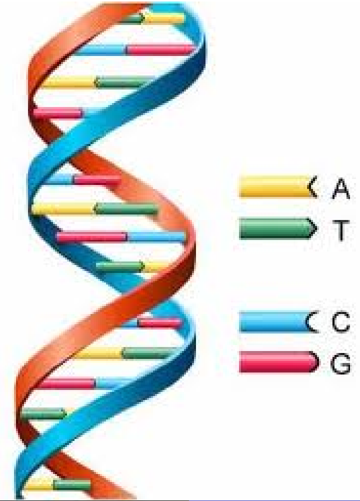
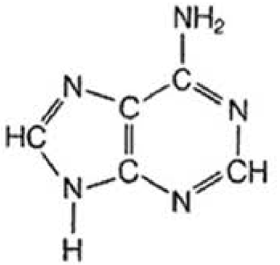


Figure : A ribosome is made of proteins

Macromolecules are made from smaller molecules

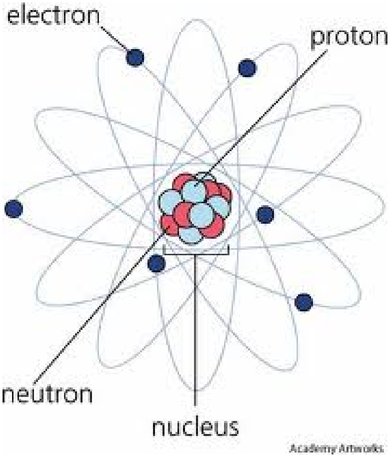


Molecules are made of atoms

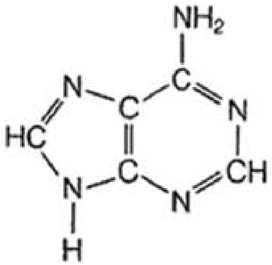


Adenine (A)

Protons, neutrons and electrons form atoms



Atoms form molecules



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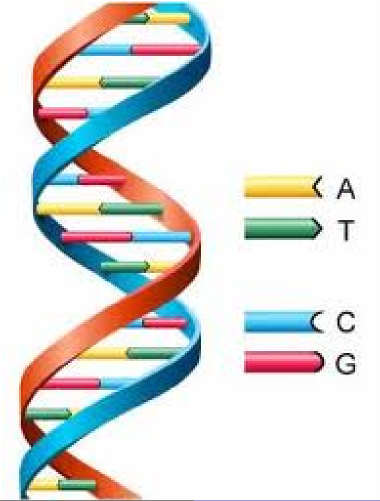
Macromolecules



Cell parts

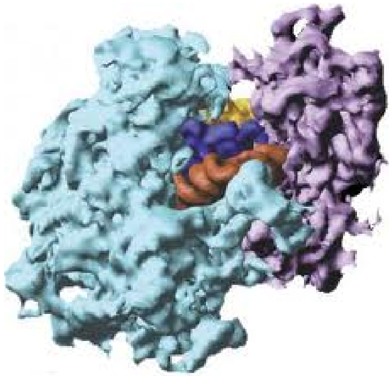
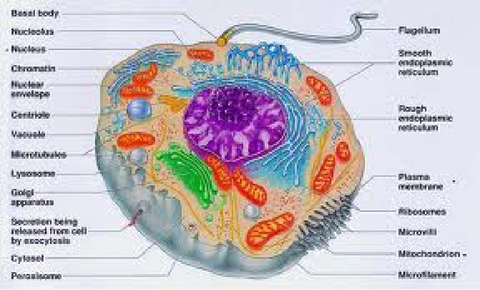
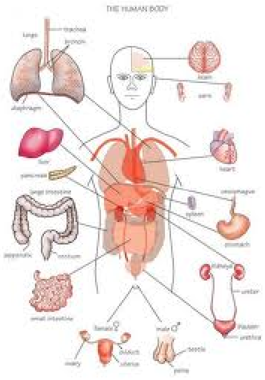


Figure : A ribosome is made of proteins

Cells


Basal body
Nucleolus
Nucleus
Chromatin
Nuclear envelope
Centriole
Vacuole
Microtubules
Lysosome
Golgi apparatus
Secretion being released from cell by exocytosis
Cytosol
Peroxisome
Flagellum
Smooth endoplasmic reticulum
Rough endoplasmic reticulum
Plasma membrane
Ribosomes
Microvilli
Mitochondrion
Microfilament

Tissues, organs and people



Counties, states and countries

Where does it end?

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- Keep on building: $a_{n+1} = a_n a_n b_n$, $b_{n+1} = a_n b_n b_n$.

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- Keep on building: $a_{n+1} = a_n a_n b_n$, $b_{n+1} = a_n b_n b_n$.
- Pattern is "self-similar", with structures of each size resembling those of the previous size. Assembly rule is the same at each stage.

Information in each tile

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- An a tile preceded by a b tile marks beginning of cluster.
- A b tile followed by a marks end of cluster.
- Label of middle tile marks type of cluster.

Recovering the hierarchy

$$\ldots aab.aab.abb.aab.abb.abb.aab.abb.abb \ldots$$

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$$\ldots \qquad a_2 \qquad\qquad b_2 \qquad\qquad b_2 \qquad \ldots.$$

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$$\ldots b_3 \ldots$$

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- If you move by less than 3^n, new n-clusters will overlap with old n-clusters.

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- If you move by less than 3^n, new n-clusters will overlap with old n-clusters.
- There is structure at arbitrarily large length scales. No movement can preserve all of these structures.

Fibonacci Sequences / Tilings

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- Two ways to think about $\sigma^{n+1}(a)$ or $\sigma^{n+1}(b)$:
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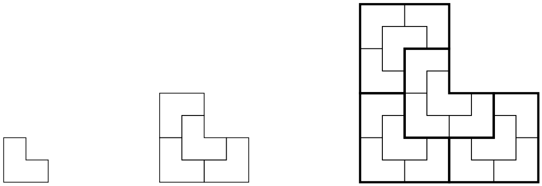
- Substitution matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Eigenvalues $2, 0$.
- Never see 3 consecutive a's or b's. Consecutive a's or b's always come from different supertiles.

Chair Tilings



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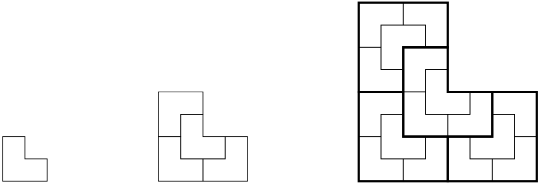
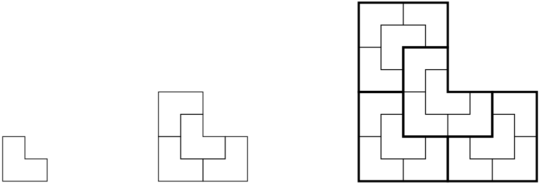


- Substitution matrix $\begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix} = 2 + R + R^{-1}$.

  Eigenvalues $4, 2, 2, 0$.

Chair Tilings



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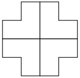
  Eigenvalues $4, 2, 2, 0$.

- Tiles in "Swiss cross"  come from four different supertiles.

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- Work with fixed finite *alphabet* of tile types (prototiles). WLOG can work with polytopes. Can have multiple prototiles with same geometry. Distinguish by labels.

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- Metric depends on choice of origin, but topology is translation-invariant. A sequence of tilings converges if its patches converge on all compact sets.
- A *tiling space* is a collection of tilings that is
  - Invariant under translation, and
  - Closed in the tiling topology.

Continuous hulls

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- $\{T - x\}$ is *orbit* of T under translation.
- Closure Ω_T of orbit is *continuous hull* of T, aka tiling space of T.
- $T' \subset \Omega_T$ iff every patch of T' is found somewhere in T.

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- $\sigma^n(t_i)$ is called an *n*-supertile of type t_i.
- A tiling T is *admissible* if every finite patch of T can be found in some supertile of arbitrary order.

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- Finite collection of tile types $t_1, \ldots, t_m$.
- Stretching factor $\lambda > 1$.
- Collection of patches $\sigma(t_i)$ with the same footprint as λt_i. σ is *substitution rule*.
- $\sigma^n(t_i)$ is called an *n*-supertile of type t_i.
- A tiling T is *admissible* if every finite patch of T can be found in some supertile of arbitrary order.
- Tiling space Ω_σ is the set of all admissible tilings.

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- When is σ injective? Surjective? A homeomorphism?

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Theorem

Under very mild assumptions*, $\sigma : \Omega_\sigma \to \Omega_\sigma$ *is surjective.*

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Recognizability and Nonperiodicity

- Substitution σ is *non-periodic* if $T \in \Omega_\sigma \wedge (T = T - x) \implies x = 0$.
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- Counterexample: $\square \to \boxplus$. Periodic and not recognizable.
- Easy theorem: Recognizability implies non-periodicity.
- Hard theorem (Mossé, Solomyak): Non-periodicity implies recognizability.

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- Relate properties of T on B_r to properties of $\sigma^{-1}(T)$ on $B_{r'}$, where $r' \approx r/\lambda$. Convergence of $\sum D/\lambda^n$ makes value of D irrelevant at large scales.
- Repeat as many times as necessary, until we are talking about small patches where we can prove things by hand.

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- Ω_T is minimal if and only if T is repetitive.
- If σ is primitive, then Ω_σ is minimal and every $T \in \Omega_\sigma$ is repetitive.

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- To resolve boundary cases, either (a) take $n \to \infty$ or (b) use collared tiles.
- Either way, $\exists!$ invariant measure on Ω_σ.

Example: Fibonacci

- $\sigma(a) = ab$, $\sigma(b) = a$, $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.
- $\lambda_{PF} = \phi := (1 + \sqrt{5})/2$.
- Left-eigenvector $(\phi, 1)$ gives lengths of tiles: Take $|a| = \phi$, $|b| = 1$.

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- Left-eigenvector $(\phi, 1)$ gives lengths of tiles: Take $|a| = \phi$, $|b| = 1$.
- Right-eigenvector $\begin{pmatrix} \phi \\ 1 \end{pmatrix}$ gives relative frequency of tiles: ϕ a's for every b.
- Density of $\sigma^n(a) = \phi^{1-n}/(2+\phi)$; density of $\sigma^n(b) = \phi^{-n}/(2+\phi)$.

Find the density of *aaba*

The pattern *aaba* is found

- Once in each $\sigma^4(a) = abaababa$,
- Zero times in $\sigma^4(b) = abaab$,
- Once in each transition $\sigma^4(a)\sigma^4(a)$.
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- Once in each transition $\sigma^4(b)\sigma^4(a)$.
- Need densities of combinations of two 4-supertiles.

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- $\rho(aaba) = 2\rho(\sigma^4(A_1)) + 2\rho(\sigma^4(A_2)) + \rho(\sigma^4(B)) = \dfrac{\phi^{-5}(2\phi+3)}{2+\phi}$.

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Eigenfunctions

- For $k \in (\mathbb{R}^n)^*$, look for functions $f : \Omega_\sigma \to \mathbb{C}$ with $f(T - x) = \exp(2\pi i k \cdot x) f(T)$.
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- If f is measurable, k is a measurable eigenvalue.
- If f is continuous, k is a continuous eigenvalue.
- **Theorem:** (Queffelec, Solomyak?) If σ is a primitive substitution, all measurable eigenvalues are continuous.

Return vectors

Generalization of "return times" for 1D dynamics.

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- E.g. for 1D substitutions, just look at displacement between successive a tiles.

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- (More general substitutions can allow λ to be in "Pisot family".)

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- Continuous spectrum is more complicated.

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- All endpoints are identified. Γ is a figure-8.

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- Two tile types give two intervals.
- aa is possible, so right endpoint of a = left endpoint of a.
- ab is possible, so right endpoint of a = left endpoint of b.
- ba is possible, so right endpoint of b = left endpoint of a.
- All endpoints are identified. Γ is a figure-8.
- σ sends vertex to itself, b loop to a loop, and a loop to a-followed-by-b.

Inverse limits

Let X be a space and $f : X \to X$ a surjective continuous map.

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- Topologize as subset of $\prod_{Z} X$. Two sequences are ϵ-close if they agree, up to ϵ, on first $1/\epsilon$ terms.
- Knowing x_n tells you $x_{n-1}, \ldots, x_1$, but not x_{n+1}. The n-th copy of X is called the *nth approximant* to $\varprojlim(X, f)$.

Dyadic solenoid

Let X be a circle. Think of n-th copy X_n as $\mathbb{R}/(2^n\mathbb{Z})$. Map f wraps circle around self twice. (See board)

Substitution Tiling Spaces are Inverse Limits

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- Point in $\varprojlim(\Gamma, f)$ is consistent instructions for placing bigger and bigger supertiles around the origin.
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- If σ is any substitution, rewriting with collared tiles makes it force the border.

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- Example: For Fibonacci, $H^1(\Gamma) = \mathbb{Z}^2$, and σ^* acts by $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Direct limit is $\mathbb{Z}^2$ since matrix is invertible.

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  - Ω_σ is uniquely ergodic. (Patch frequencies are well-defined.)
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