Cellular automata, tilings and (un)computability

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Topics of the four lectures

(1) Wang tiles: aperiodicity and undecidability

(2) Tutorial on Cellular Automata

(3) From tiles to cellular automata

(4) Snakes and tiles

Wang tiles

Wang tile: a unit square tile with colored edges.

Tile set T: a finite collection of such tiles.

Valid tiling: an assignment

$$\mathbb{Z}^2 \longrightarrow T$$

of tiles on infinite square lattice so that the abutting edges of adjacent tiles have the same color.

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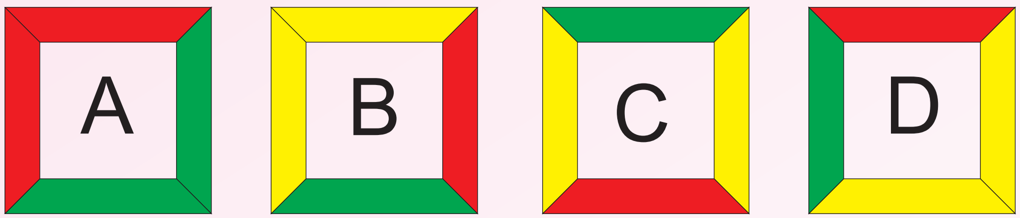
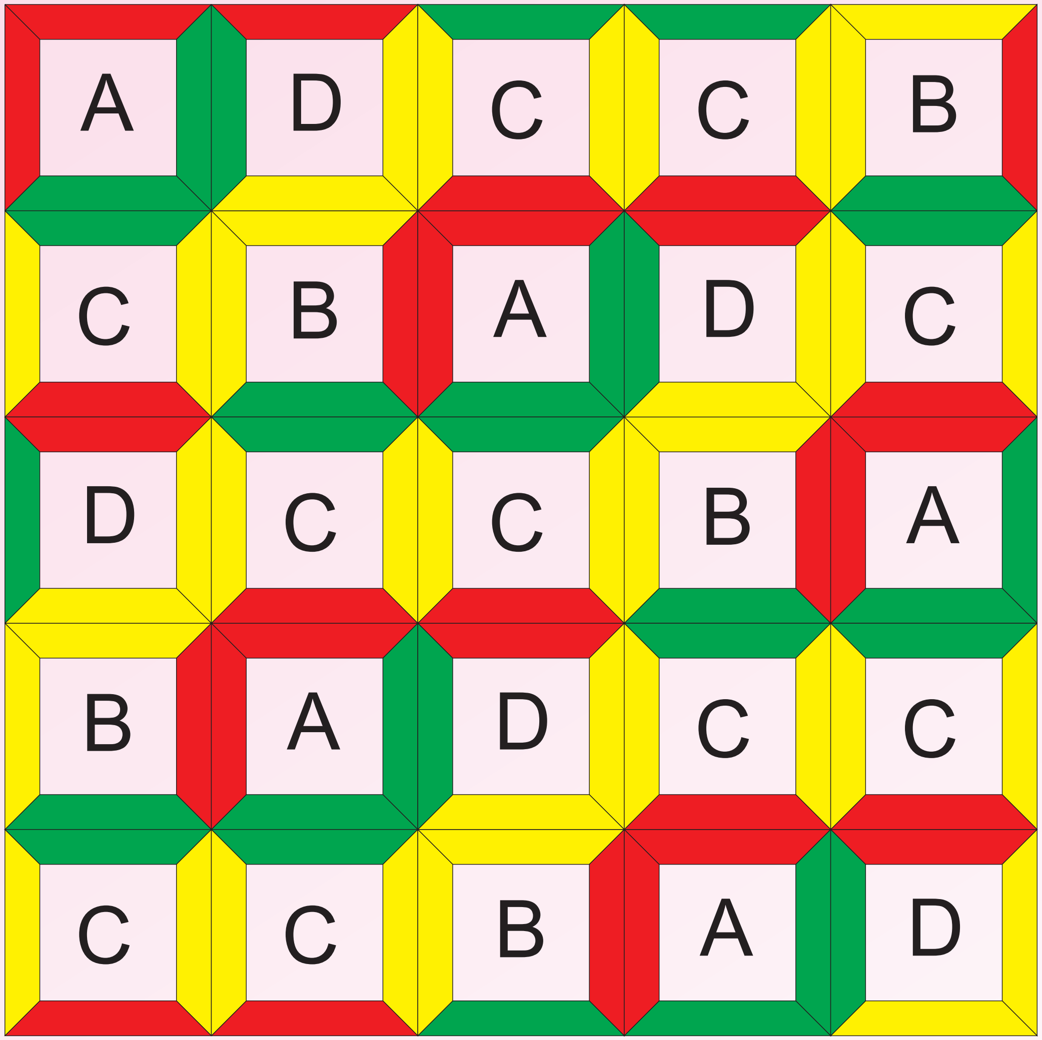


With copies of the given four tiles we can properly tile a 5×5 square…



… and since the colors on the borders match this square can be repeated to form a valid periodic tiling of the plane.

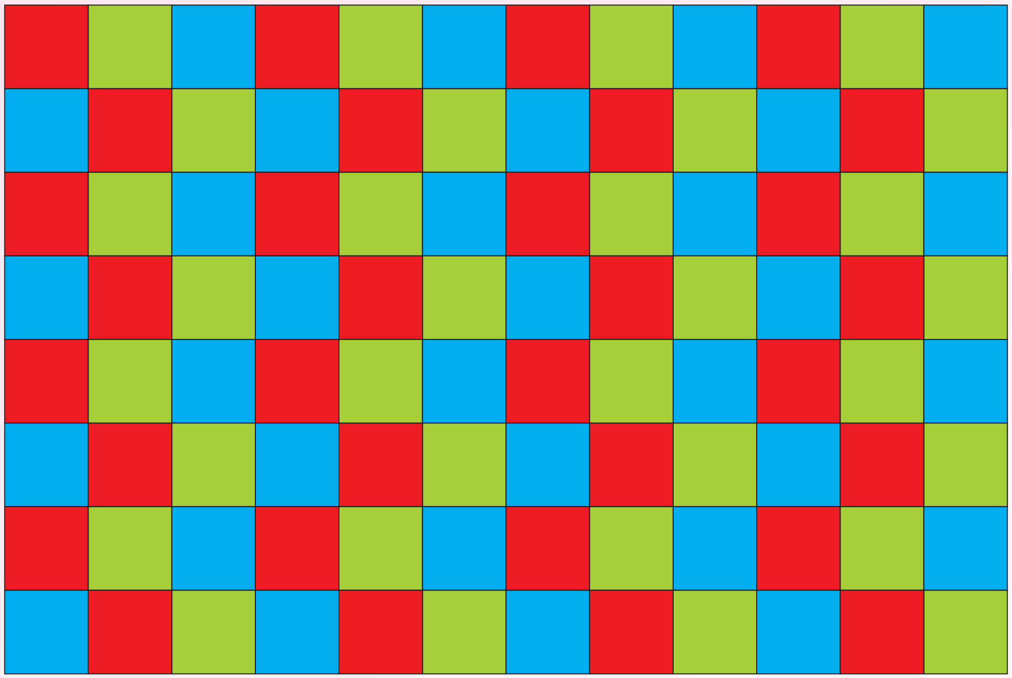
The **tiling problem** of Wang tiles is the decision problem to determine if a given finite set of Wang tiles admits a valid tiling of the plane.

Theorem (R.Berger 1966): The tiling problem of Wang tiles is undecidable.

Aperiodicity

A tiling is called **periodic** if it is invariant under some non-zero translation of the plane.

A Wang tile set that admits a periodic tiling also admits a doubly periodic tiling: a tiling with a horizontal and a vertical period:



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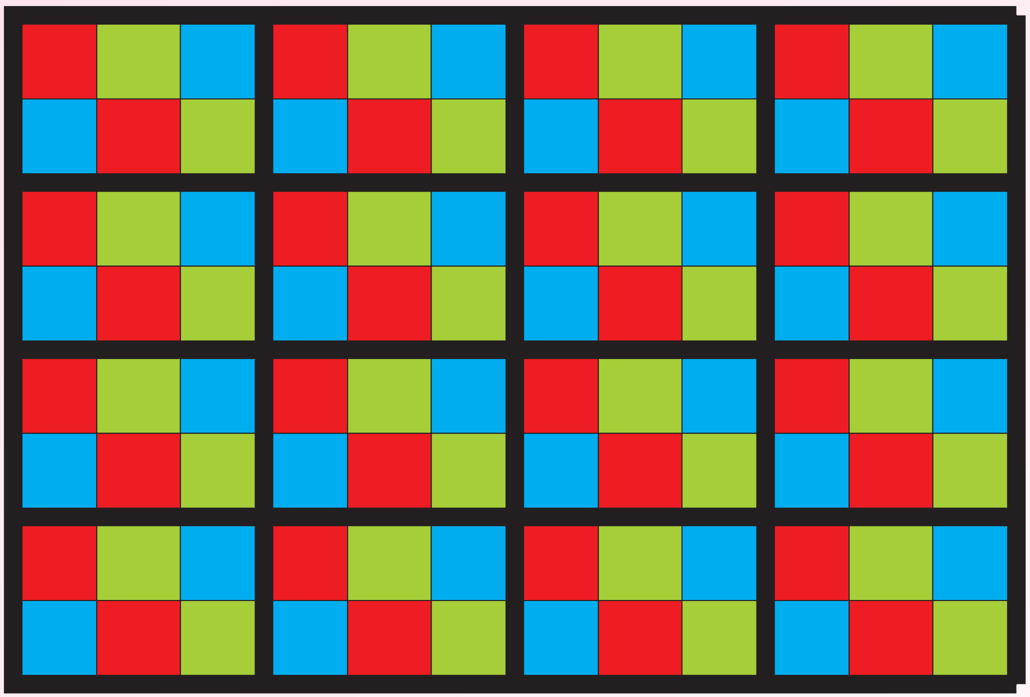

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$$T \text{ admits tiling } \Longrightarrow T \text{ admits periodic tiling.}$$

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There is a tile set that admits a tiling but does not admit periodic tilings.

Such tile sets are called **aperiodic**.

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Such tile sets are called **aperiodic**.

Berger's aperiodic tile set contained 20,426 tiles.

In this talk: 14 tiles, simple proof of aperiodicity.

Smallest possible: 11 tiles (by E. Jeandel and M. Rao)

Remark: If Wang's conjecture had been true then the tiling problem would be decidable:

Try all possible tilings of larger and larger rectangles until either

(a) **a rectangle is found that can not be tiled (so no tiling of the plane exists)**, or

(b) **a tiling of a rectangle is found that can be repeated periodically to form a periodic tiling.**

Only aperiodic tile sets fail to reach either (a) or (b)...

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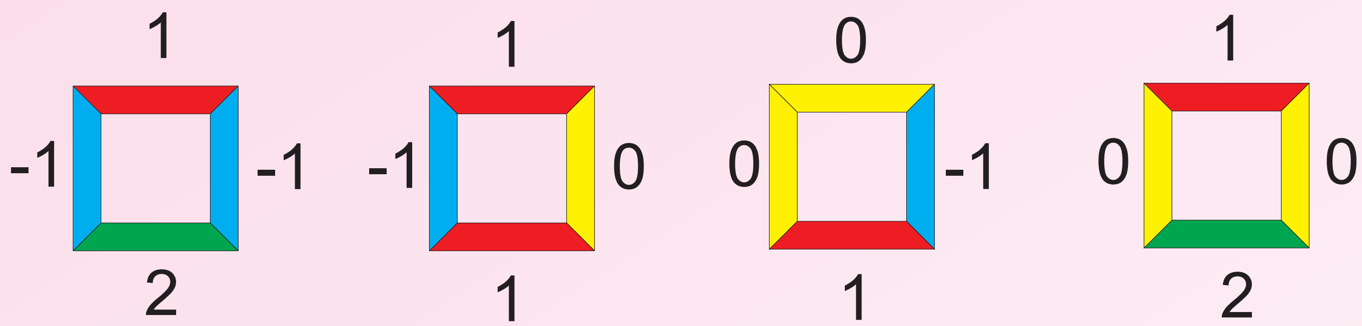
Any undecidability proof of the tiling problem must contain (explicitly or implicitly) a construction of an aperiodic tile set.

14 tile aperiodic set

The colors in our Wang tiles are real numbers, for example



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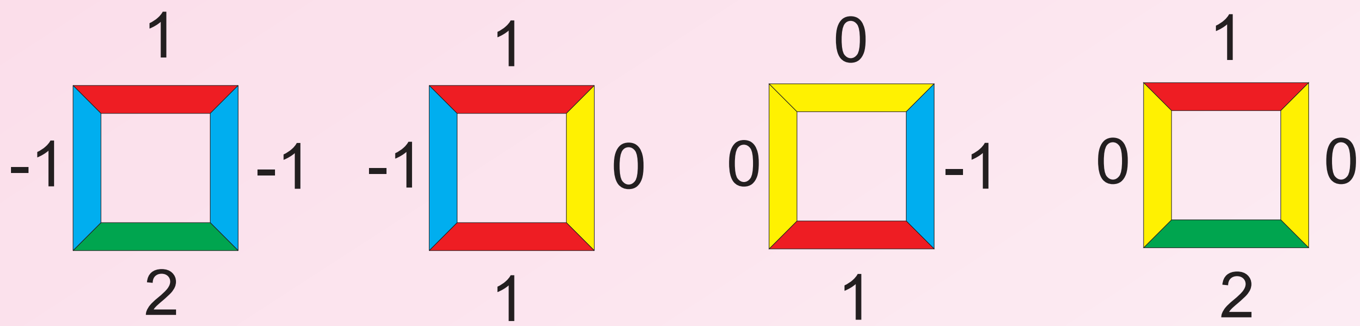
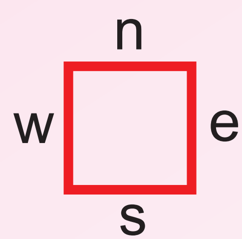


We say that tile



multiplies by number $q \in \mathbb{R}$ if

$$qn + w = s + e.$$

(The **"input"** n comes from the north, and the **"carry-in"** w from the west is added to the product qn. The result is split between the **"output"** s to the south and the **"carry-out"** e to the east.)

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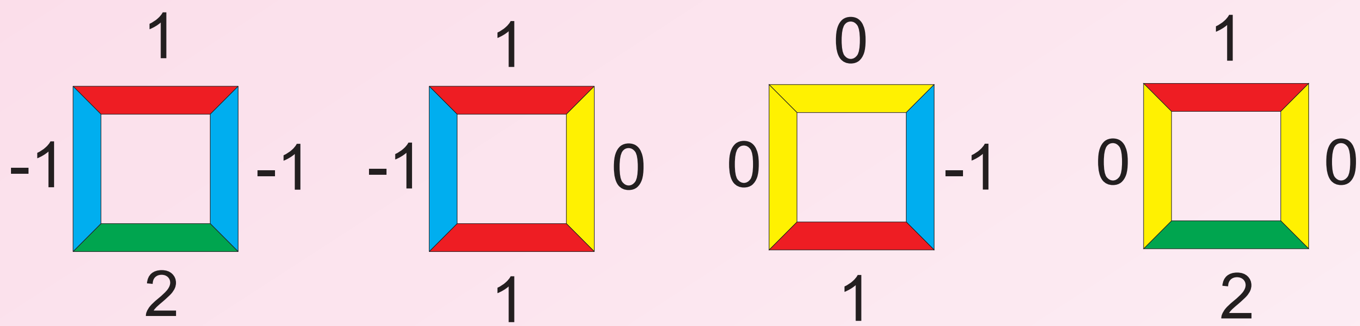

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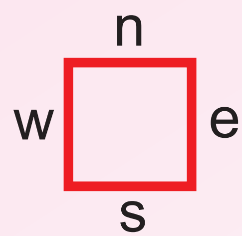
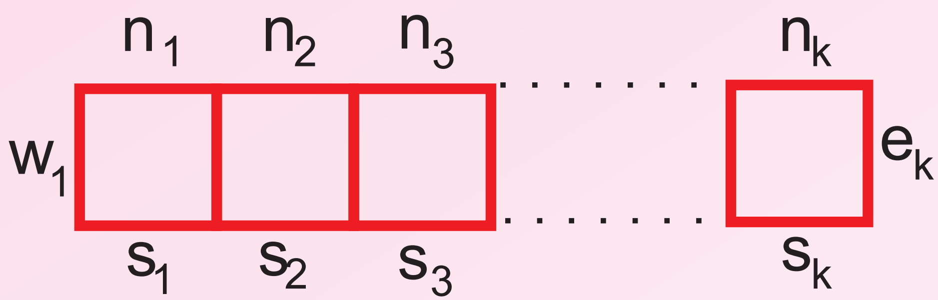


multiplies by number $q \in \mathbb{R}$ if

$$qn + w = s + e.$$

The four sample tiles above all multiply by $q = 2$.

Suppose we have a correctly tiled horizontal segment where all tiles multiply by the same q.



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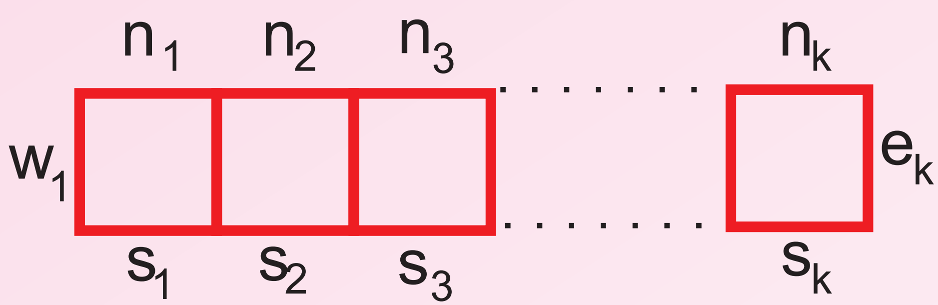

Adding up

$$\begin{aligned} qn_1 + w_1 &= s_1 + e_1 \\ qn_2 + w_2 &= s_2 + e_2 \\ &\vdots \\ qn_k + w_k &= s_k + e_k, \end{aligned}$$

taking into account that $e_i = w_{i+1}$ gives

$$\mathbf{q(n_1 + n_2 + \ldots + n_k) + w_1 = (s_1 + s_2 + \ldots + s_k) + e_k.}$$

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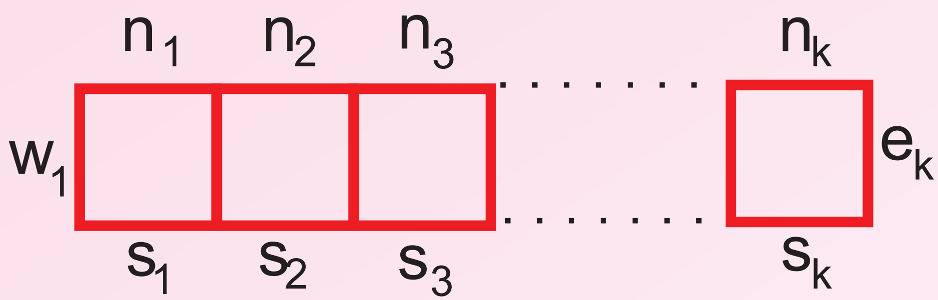
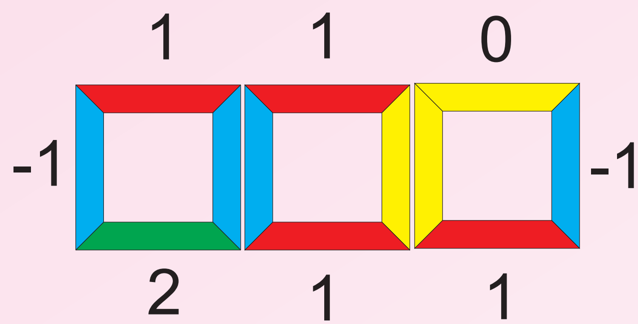


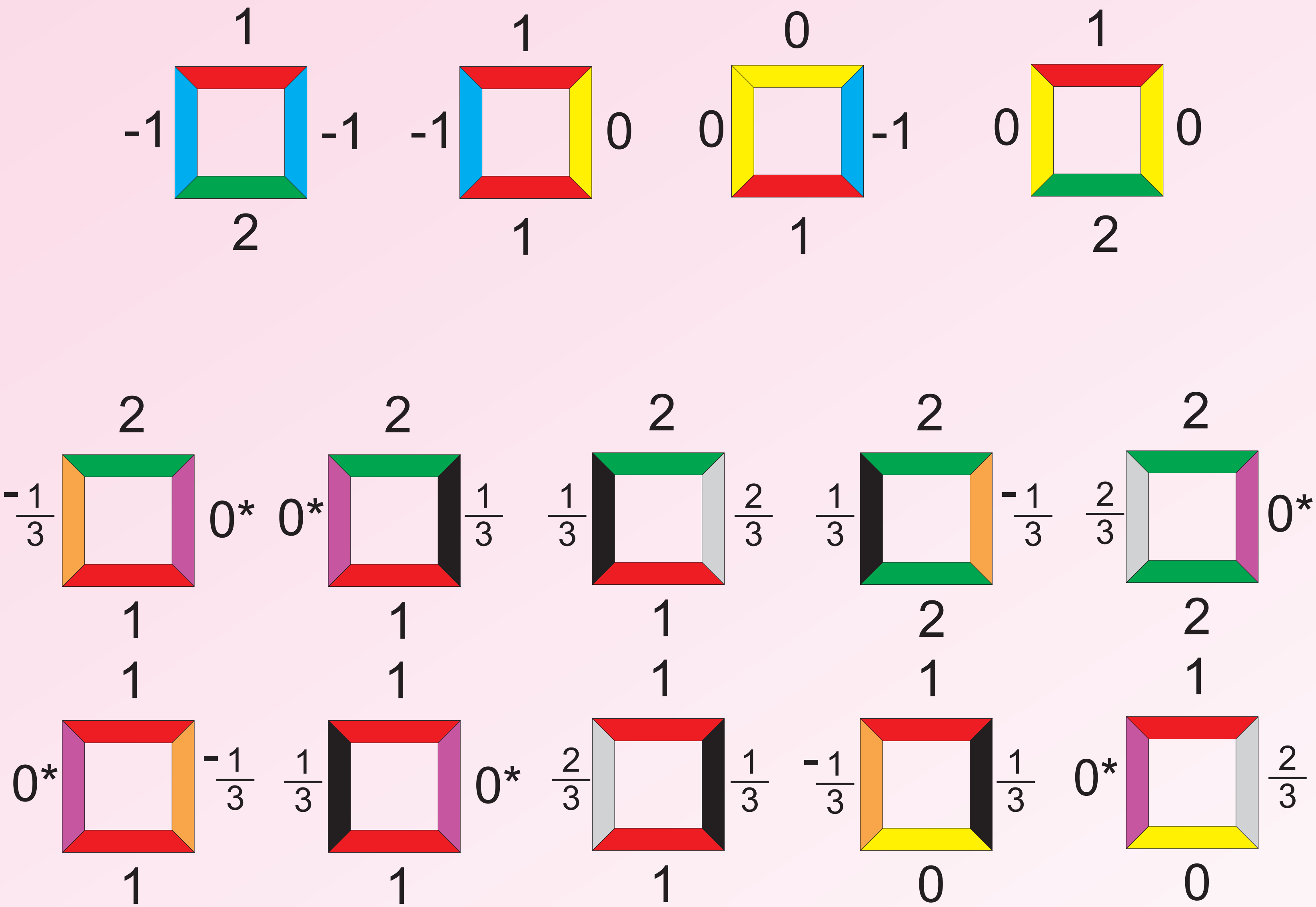
If, moreover, the segment begins and ends in the same color $(w_1 = e_k)$ then

$$\mathbf{q(n_1 + n_2 + \ldots + n_k) = (s_1 + s_2 + \ldots + s_k).}$$

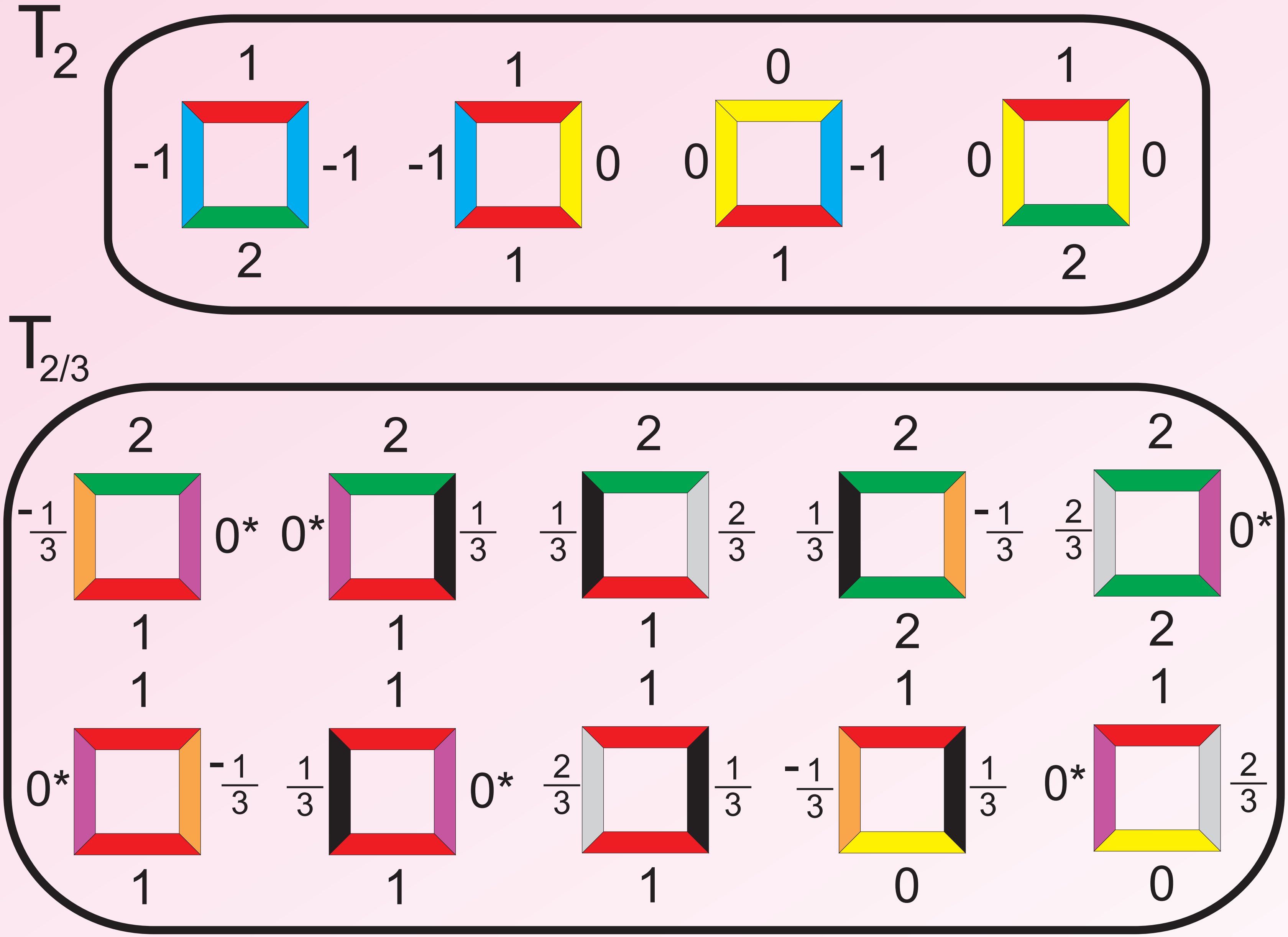
For example, our sample tiles that multiply by $q = 2$ admit the segment



The sum of the bottom labels is twice the sum of the top labels.



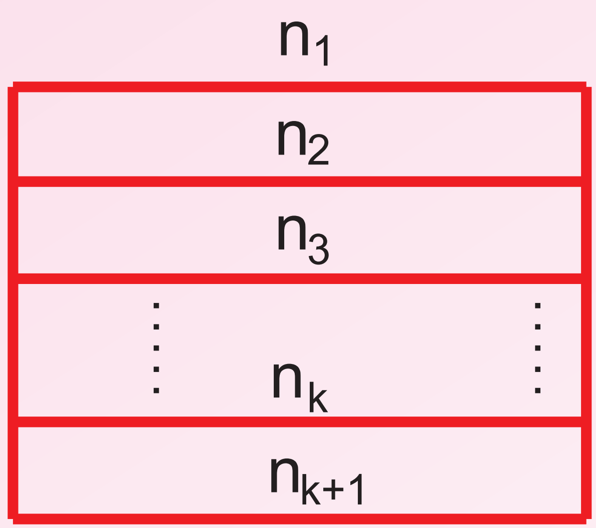
An aperiodic 14 tile set: four tiles that all multiply by 2, and 10 tiles that all multiply by $\frac{2}{3}$.



Let us call these two tile sets T_2 and $T_{2/3}$. Vertical colors are disjoint, so every horizontal row of a tiling comes entirely from one of the two sets.

No periodic tiling exists.

Suppose the contrary: A rectangle can be tiled whose top and bottom rows match and left and right sides match.



Denote by n_i the sum of the numbers on the i'th row. The tiles of the i'th row multiply by $q_i \in \{2, \frac{2}{3}\}$.

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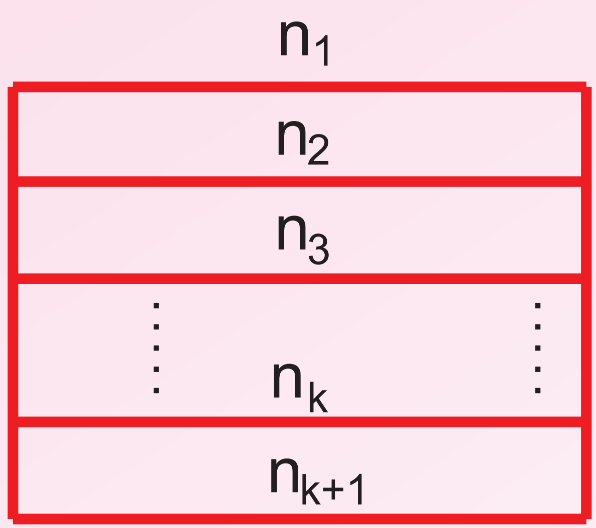

Denote by n_i the sum of the numbers on the i'th row. The tiles of the i'th row multiply by $q_i \in \{2, \frac{2}{3}\}$.

Then $n_{i+1} = q_i n_i$, for all i.

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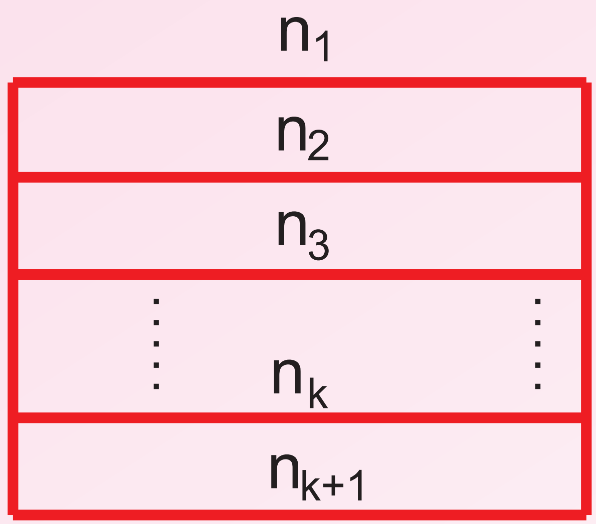

So we have $n_1 q_1 q_2 q_3 \ldots q_k = n_{k+1}$

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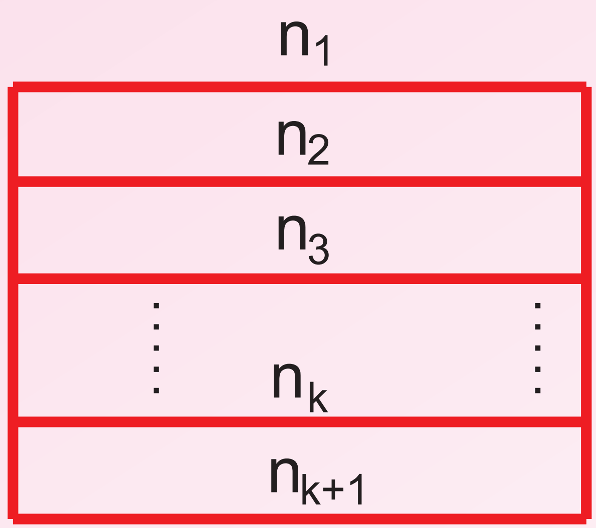

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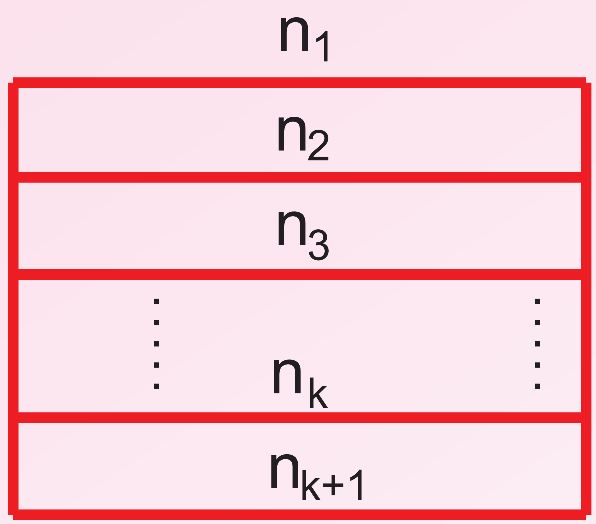

So we have $n_1 q_1 q_2 q_3 \ldots q_k = n_{k+1} = n_1$.

Clearly $n_1 > 0$, so we have $q_1 q_2 q_3 \ldots q_k = 1$.

But this is not possible since 2 and 3 are relatively prime: No product of numbers 2 and $\frac{2}{3}$ can equal 1.

Next step: Proof that a valid tiling of the plane exists.

We use **sturmian** or **balanced** representations of real numbers as bi-infinite sequences of two closest integers.

The representation of any $\alpha \in \mathbb{R}$ is the sequence $B(\alpha)$ whose k'th element is

$$B_k(\alpha) = \lfloor k\alpha \rfloor - \lfloor (k-1)\alpha \rfloor.$$

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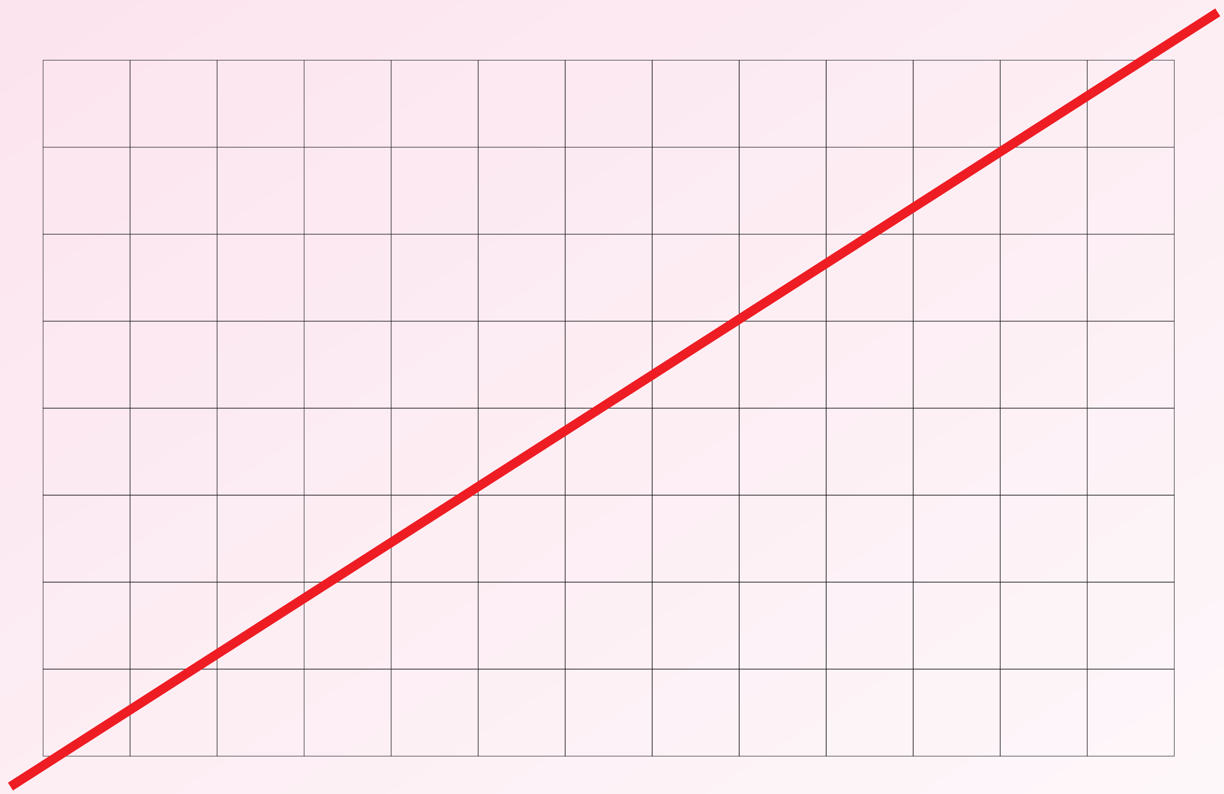

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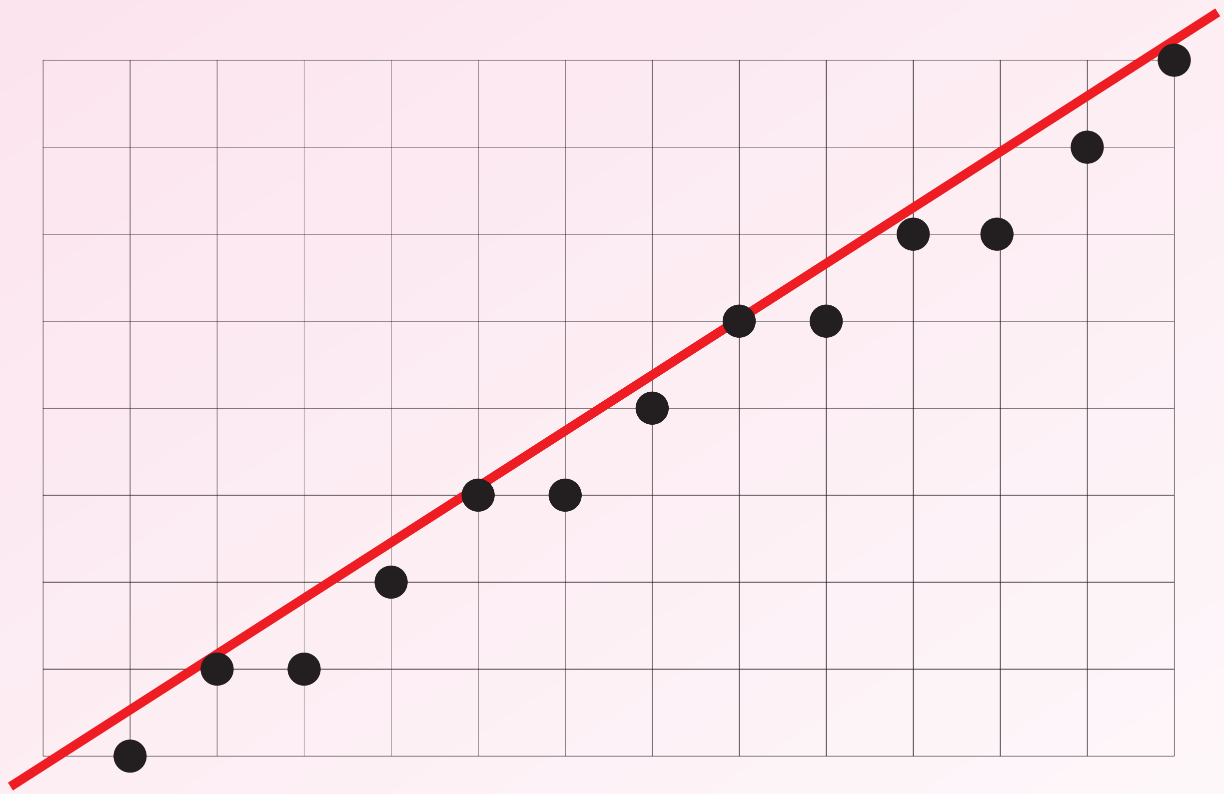

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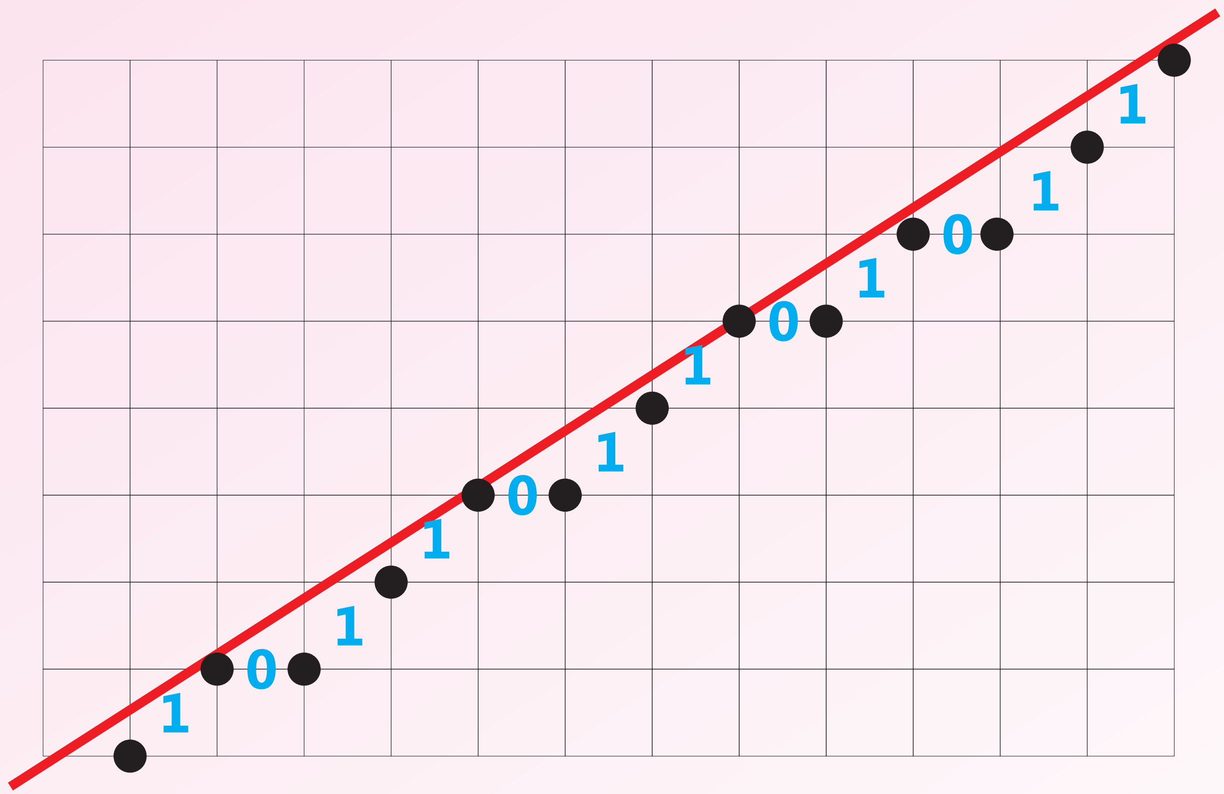

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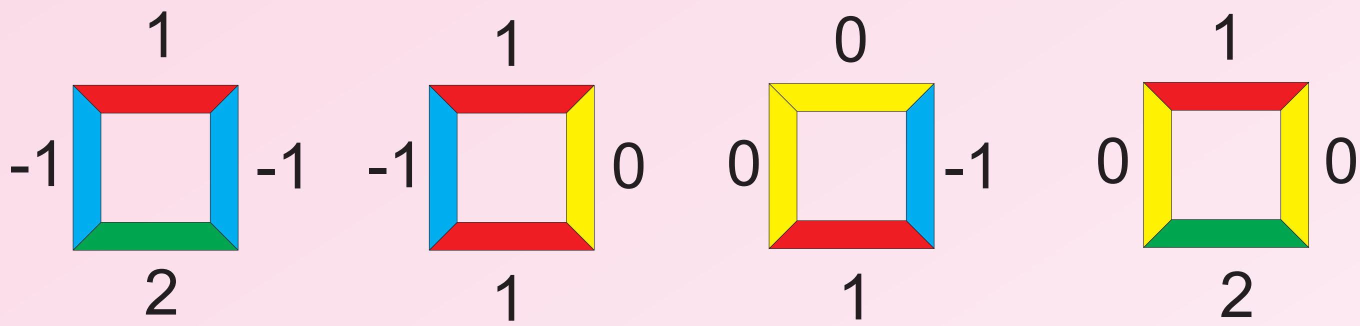
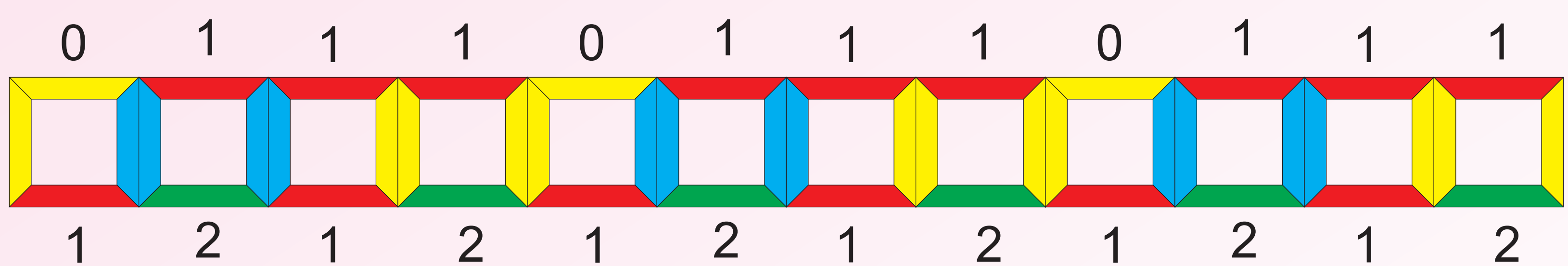
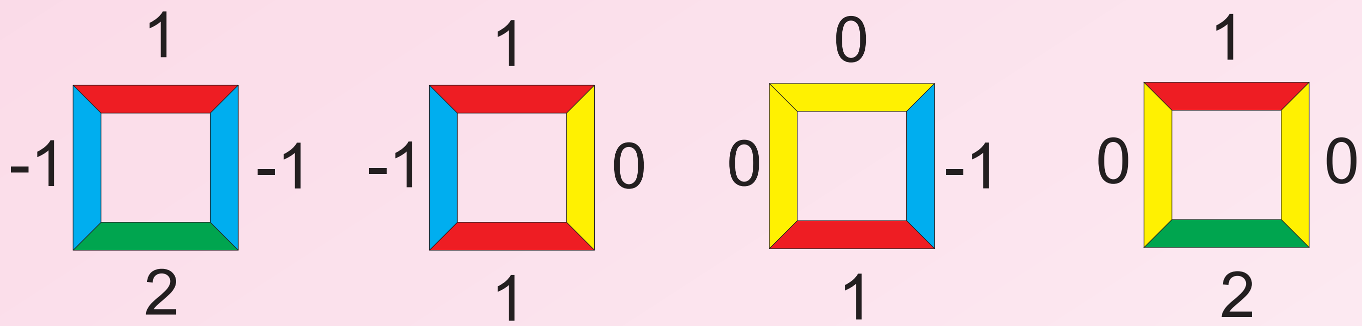


The first tile set T_2 admits a tiling of every infinite horizontal strip whose top and bottom labels read $B(\alpha)$ and $B(2\alpha)$, for all $\alpha \in \mathbb{R}$ satisfying

$$
\begin{array}{ccccc}
0 & \leq & \alpha & \leq & 1, \text{ and} \\
1 & \leq & 2\alpha & \leq & 2.
\end{array}
$$

For example, with $\alpha = \frac{3}{4}$:





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$$\left.\begin{array}{ccccl} 0 & \leq & \alpha & \leq & 1, \text{ and} \\ 1 & \leq & 2\alpha & \leq & 2. \end{array}\right\} \Longleftrightarrow \frac{1}{2} \leq \alpha \leq 1$$

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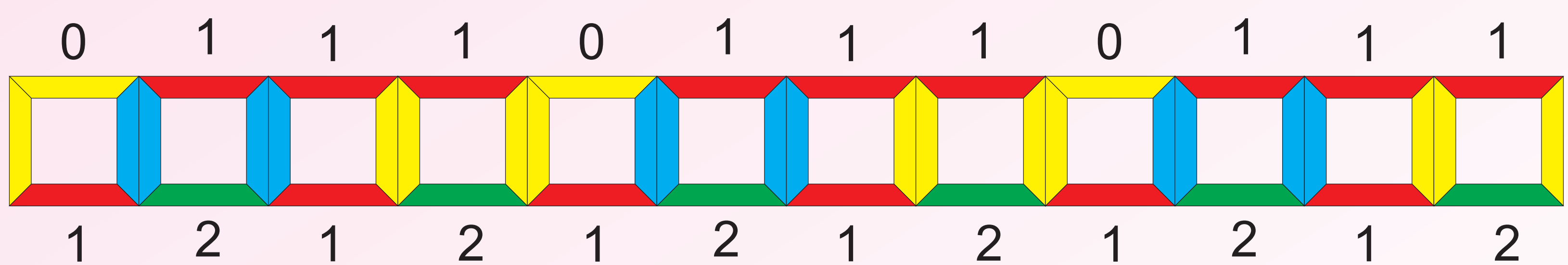
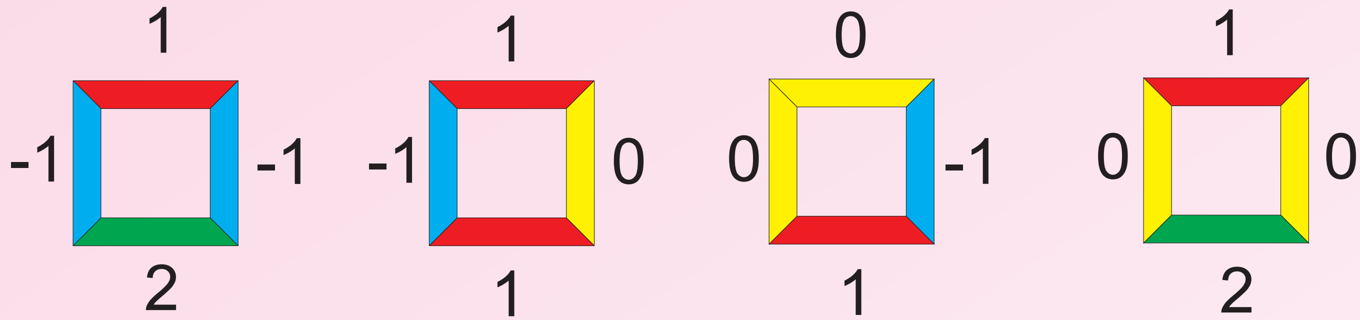
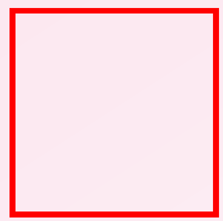




This is guaranteed by including in the tile set for every $\frac{1}{2} \leq \alpha \leq 1$ and every $k \in \mathbb{Z}$ the following tile

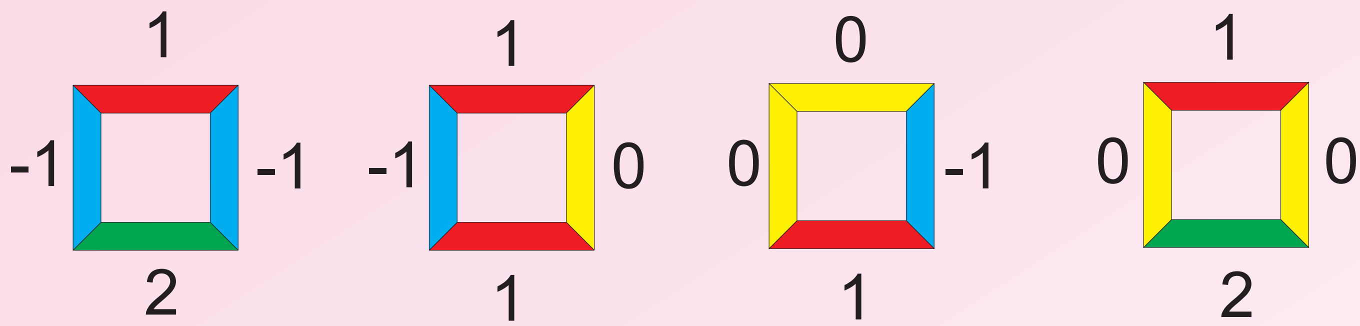
$$B_k(\alpha)$$

$$2\lfloor (k-1)\alpha \rfloor - \lfloor 2(k-1)\alpha \rfloor$$



$$2\lfloor k\alpha \rfloor - \lfloor 2k\alpha \rfloor$$

$$B_k(2\alpha)$$



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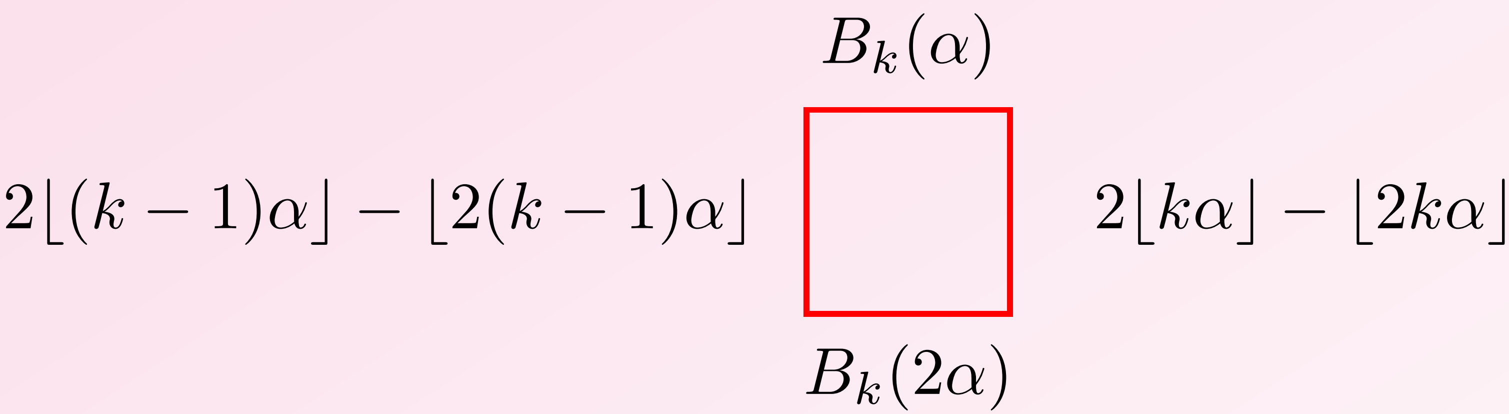
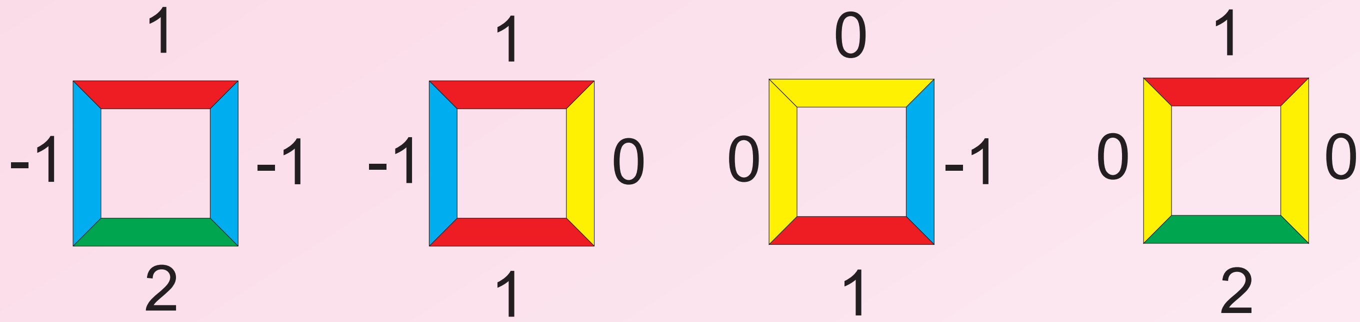


(1) For fixed α the tiles for consecutive $k \in \mathbb{Z}$ match so that a horizontal row can be formed whose top and bottom labels read the balanced representations of α and 2α, respectively.



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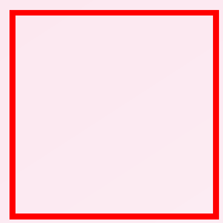
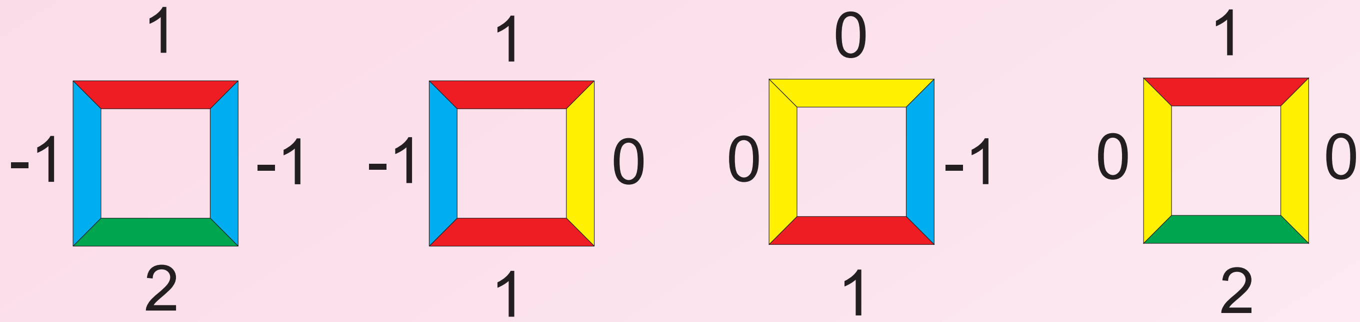


$$2\lfloor k\alpha \rfloor - \lfloor 2k\alpha \rfloor$$

$$B_k(2\alpha)$$

(2) A direct calculation shows that the tile multiplies by 2, that is,

$$2n + w = s + e.$$



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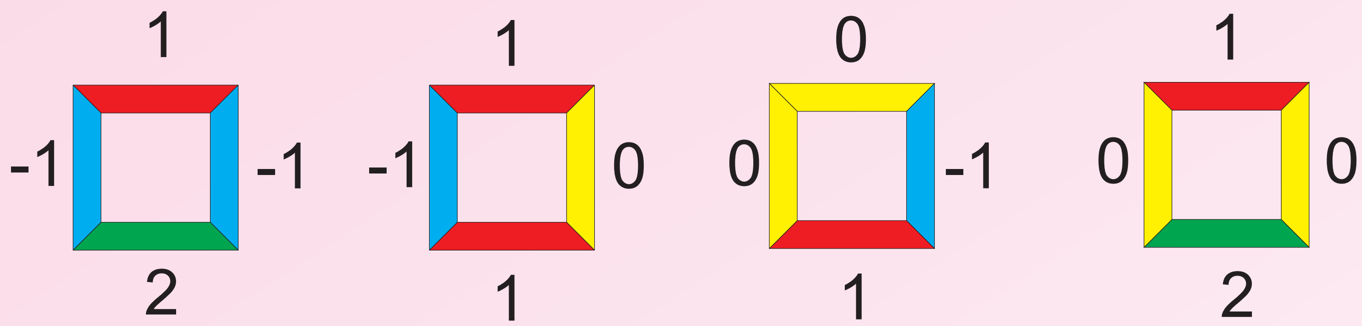
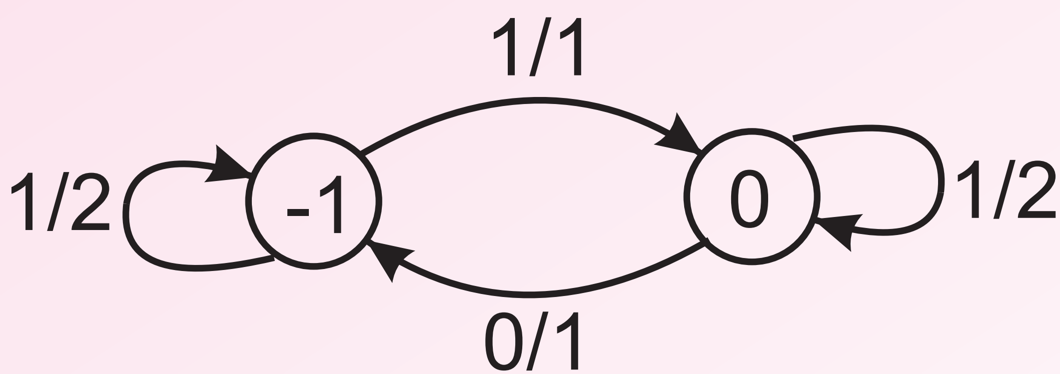
(3) There are only finitely many such tiles, even though there are infinitely many $k \in \mathbb{Z}$ and α. These are the four tiles in T_2.



The four tiles can be also interpreted as edges of a **finite state transducer** whose states are the vertical colors and input/output symbols of transitions are the top and the bottom colors:



A tiling of an infinite horizontal strip is a **bi-infinite path** whose input symbols and output symbols read the top and bottom colors of the strip. We have enough transitions to allow the transducer to convert $B(\alpha)$ into $B(2\alpha)$.

An analogous construction can be done for **any rational multiplier q**. We can construct the following tiles for all $k \in \mathbb{Z}$ and all α in the domain interval:

$$B_k(\alpha)$$

$$q\lfloor (k-1)\alpha \rfloor - \lfloor q(k-1)\alpha \rfloor \qquad \boxed{} \qquad q\lfloor k\alpha \rfloor - \lfloor qk\alpha \rfloor$$

$$B_k(q\alpha)$$

The tiles multiply by q, and they admit a tiling of a horizontal strip whose top and bottom labels read $B(\alpha)$ and $B(q\alpha)$.

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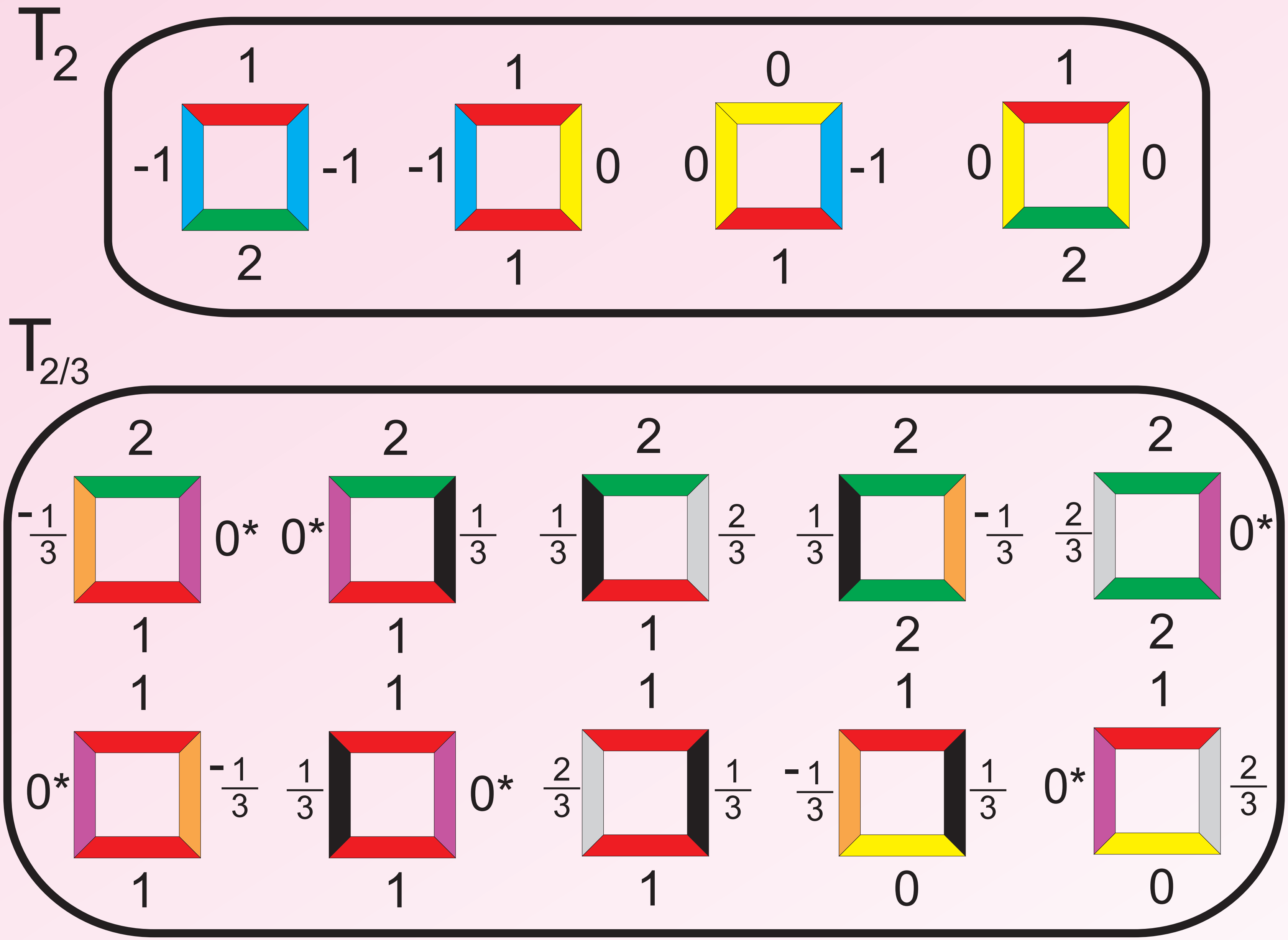
Our second tile set $T_{2/3}$ was constructed in this way for $q = \frac{2}{3}$ and interval $1 \leq \alpha \leq 2$.



T_2
1 / -1 / -1 / 2
1 / -1 / 0 / 1
0 / 0 / -1 / 1
1 / 0 / 0 / 2
$T_{2/3}$
2 / $-\frac{1}{3}$ / 0* / 1
2 / 0* / $\frac{1}{3}$ / 1
2 / $\frac{1}{3}$ / $\frac{2}{3}$ / 1
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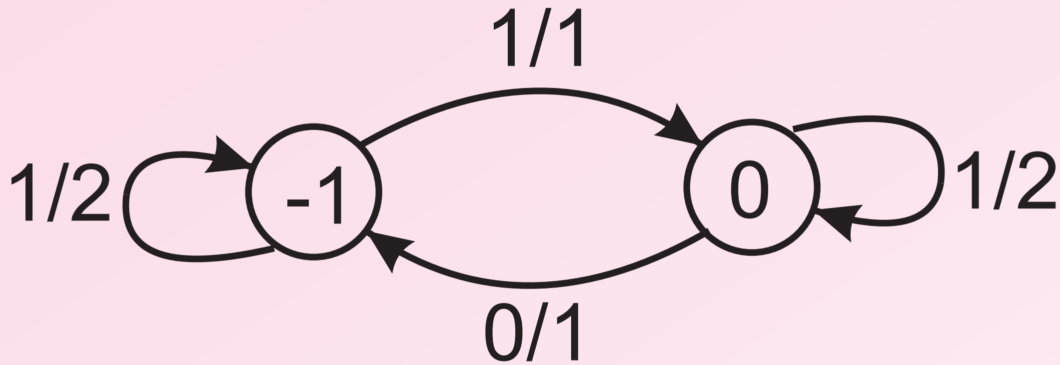
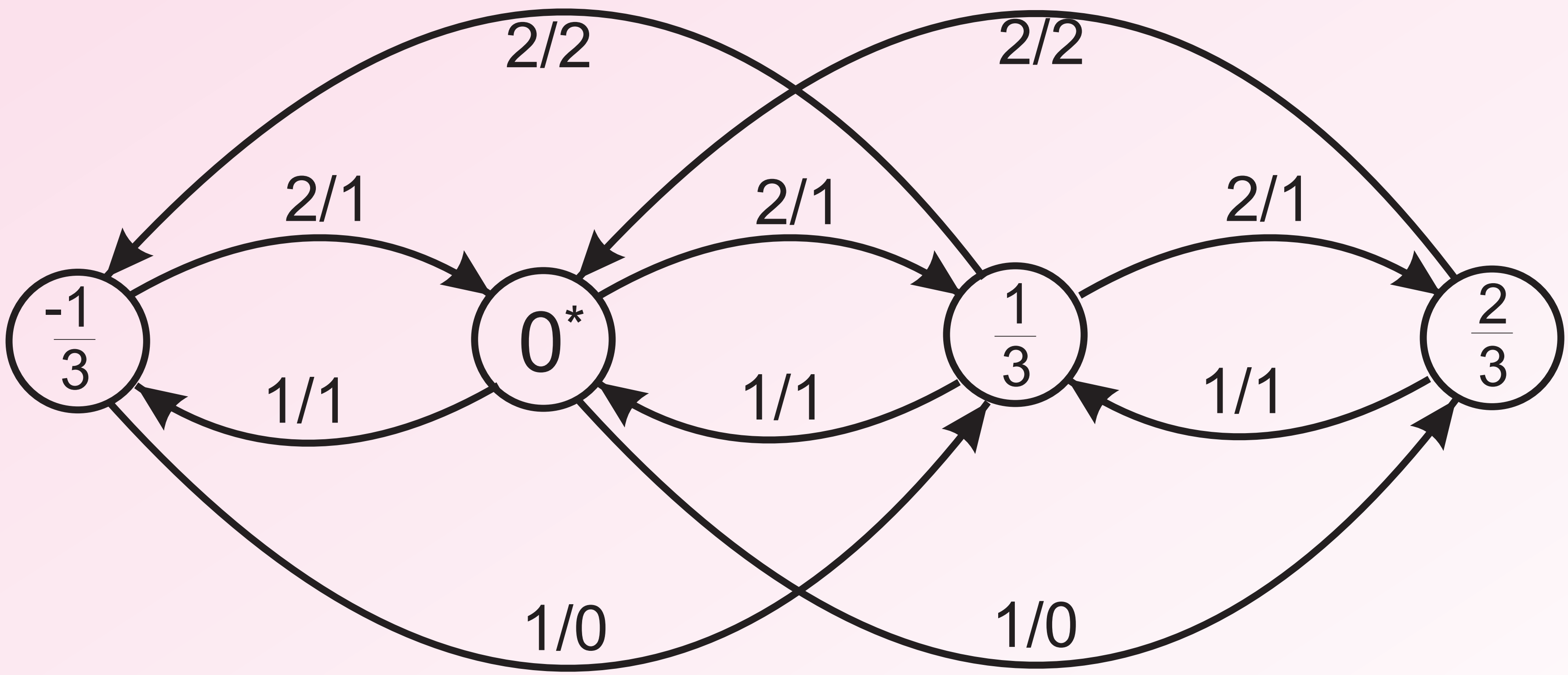
1 / 0* / $-\frac{1}{3}$ / 1
1 / $\frac{1}{3}$ / 0* / 1
1 / $\frac{2}{3}$ / $\frac{1}{3}$ / 1
1 / $-\frac{1}{3}$ / $\frac{1}{3}$ / 0
1 / 0* / $\frac{2}{3}$ / 0


1/1
1/2
-1
0
1/2
0/1

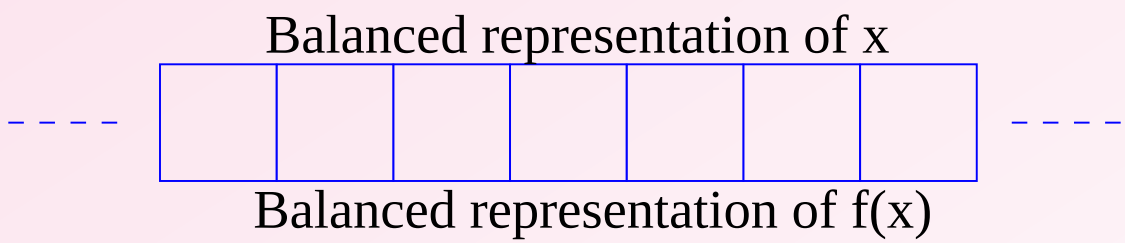

2/2
2/2
2/1
2/1
2/1
-1/3
0*
1/3
2/3
1/1
1/1
1/1
1/0
1/0

The tiles admit valid tilings of the plane that simulate iterations of the piecewise linear dynamical system

$$f : [\frac{1}{2}, 2] \longrightarrow [\frac{1}{2}, 2]$$

where

$$f(x) = \begin{cases} 2x, & \text{if } x \leq 1, \text{ and} \\ \frac{2}{3}x, & \text{if } x > 1. \end{cases}$$



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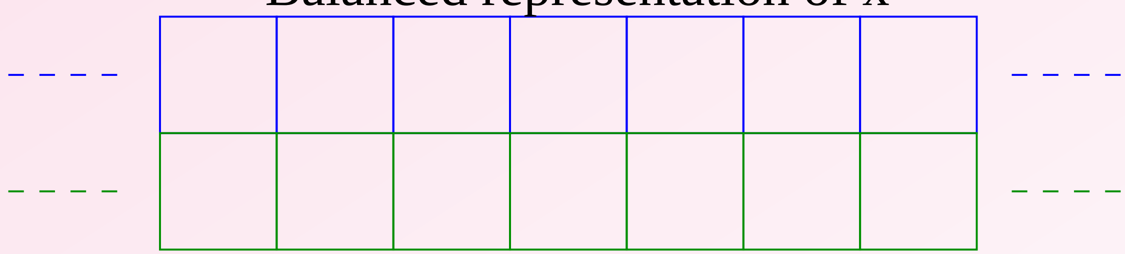
Balanced representation of x



Balanced representation of $f^2(x)$

The tiles admit valid tilings of the plane that simulate iterations of the piecewise linear dynamical system

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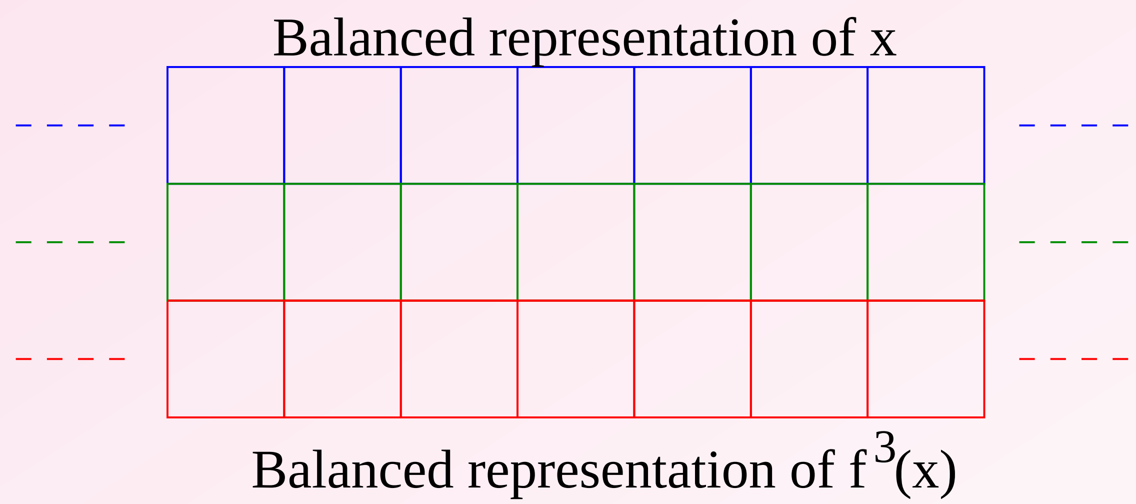

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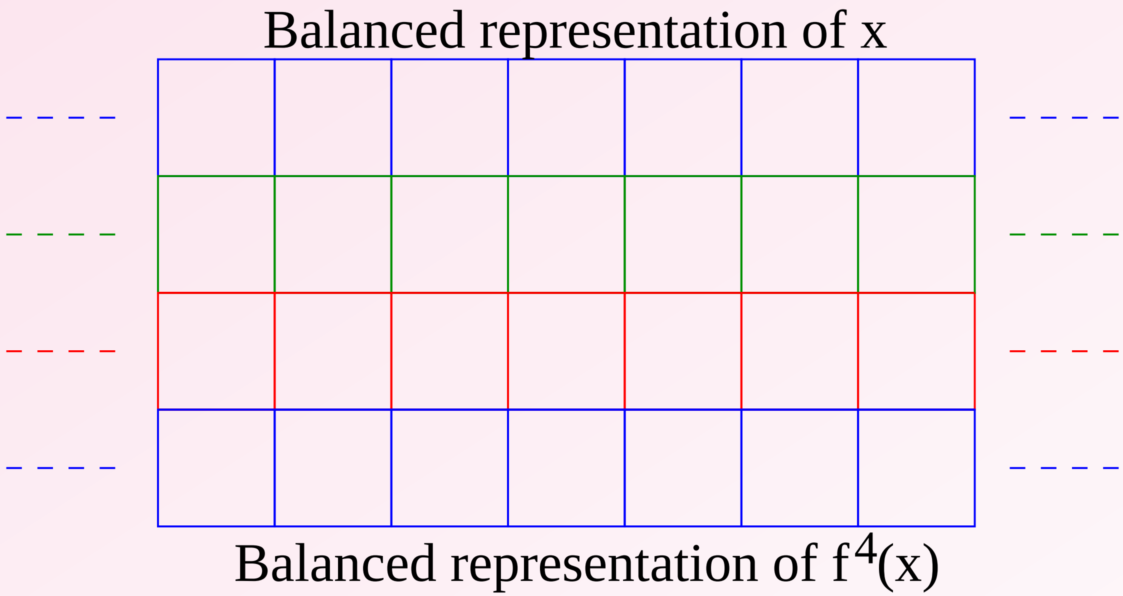

Undecidability of the tiling problem

Similar construction can be effectively carried out for any piecewise linear function on a union of finite intervals of $\mathbb{R}$, as long as the multiplications are with rational numbers q.

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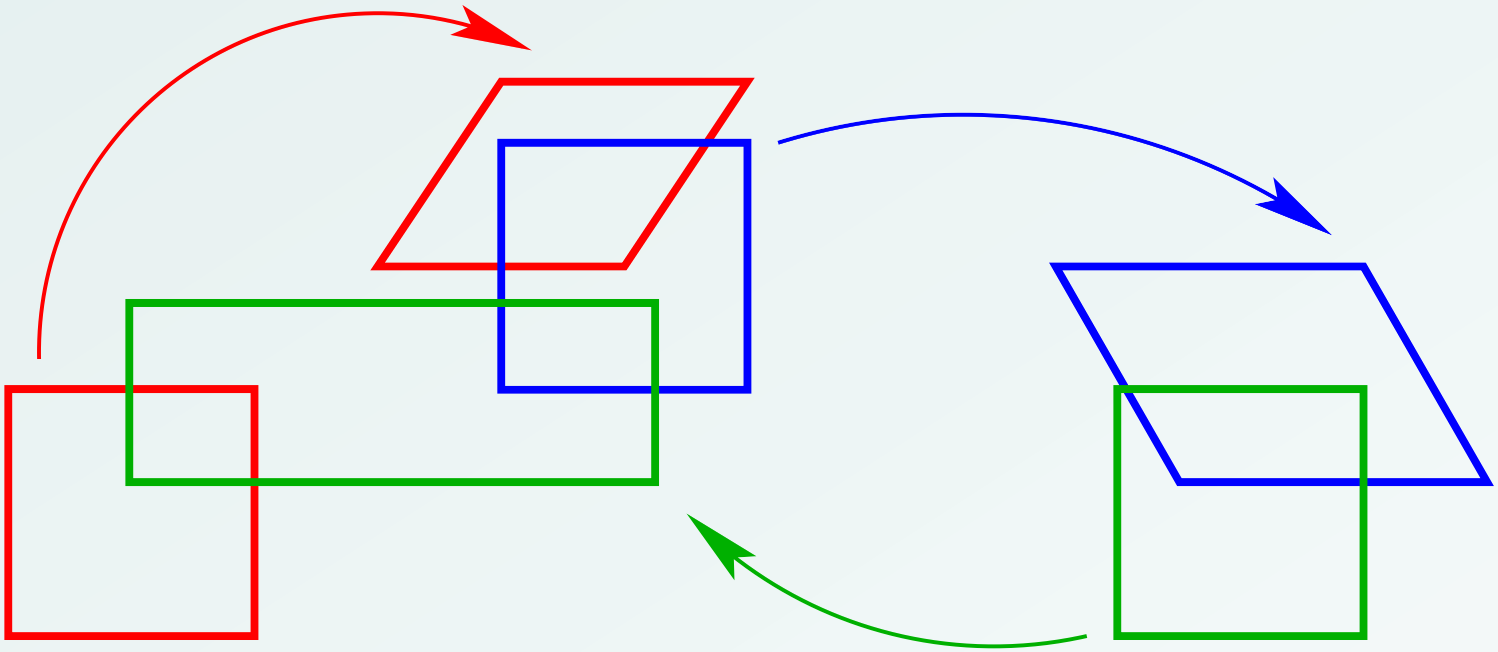
In order to prove **undecidability results** concerning tilings we want to simulate more complex dynamical systems that can carry out Turing computations.

We generalize the construction in two ways:

- from linear maps to affine maps, and
- from $\mathbb{R}$ to $\mathbb{R}^2$, (or $\mathbb{R}^d$ for any d).

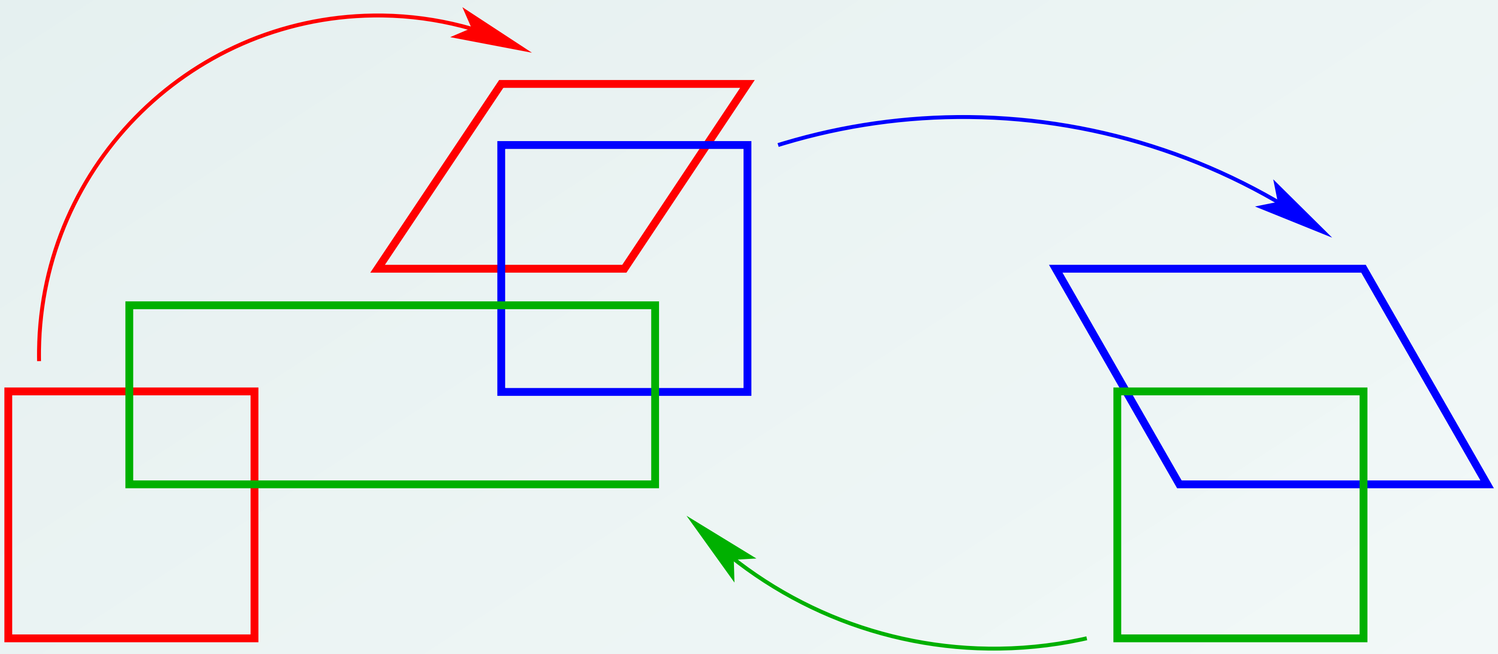
Immortality of piecewise affine maps



Consider a system of finitely many pairs (U_i, f_i) where

- U_i are disjoint unit squares of the plane with integer corners,
- f_i are affine transformations with rational coefficients.

Square U_i serves as the domain where f_i may be applied.



The system determines a function

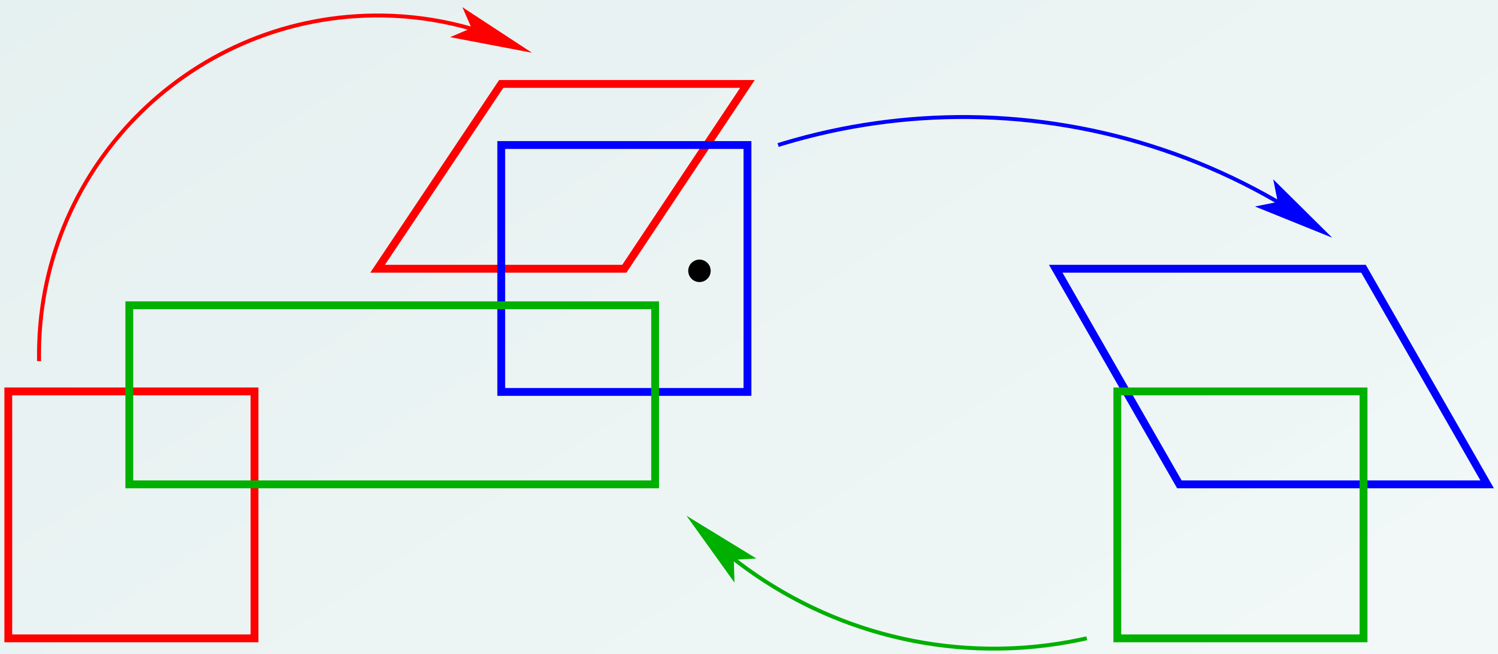
$$f : D \longrightarrow \mathbb{R}^2$$

whose domain is

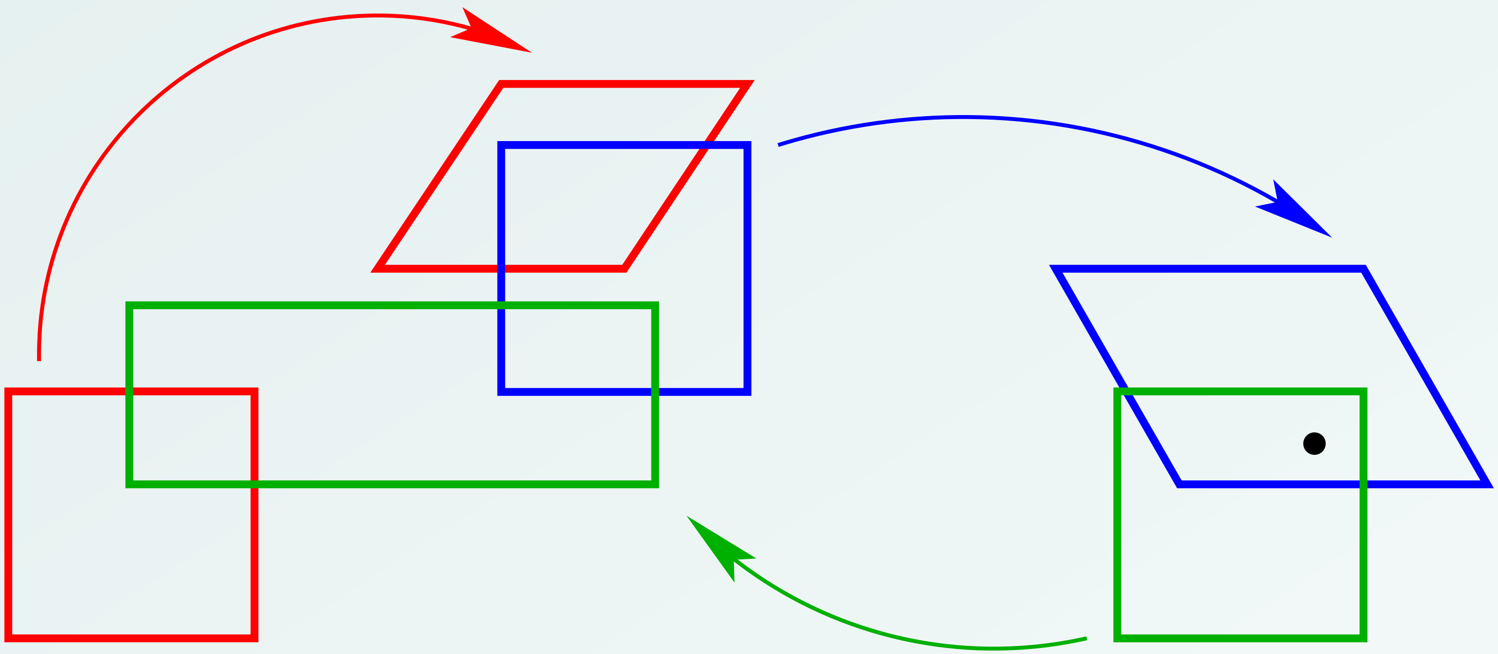
$$D = \bigcup_i U_i$$

and

$$f(\vec{x}) = f_i(\vec{x}) \text{ for all } \vec{x} \in U_i.$$



The orbit of $\vec{x} \in D$ is the iteration of f starting at point $\vec{x}$. The iteration can be continued as long as the point remains in the domain D.



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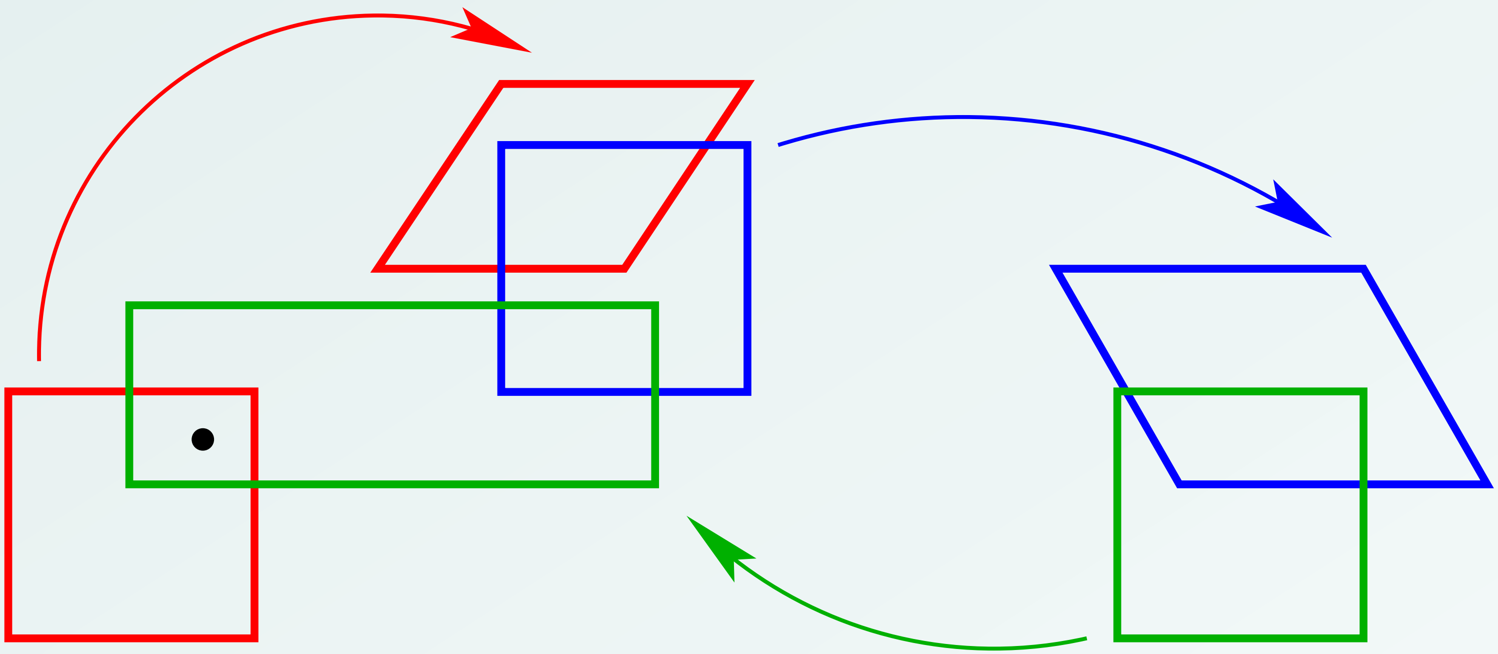

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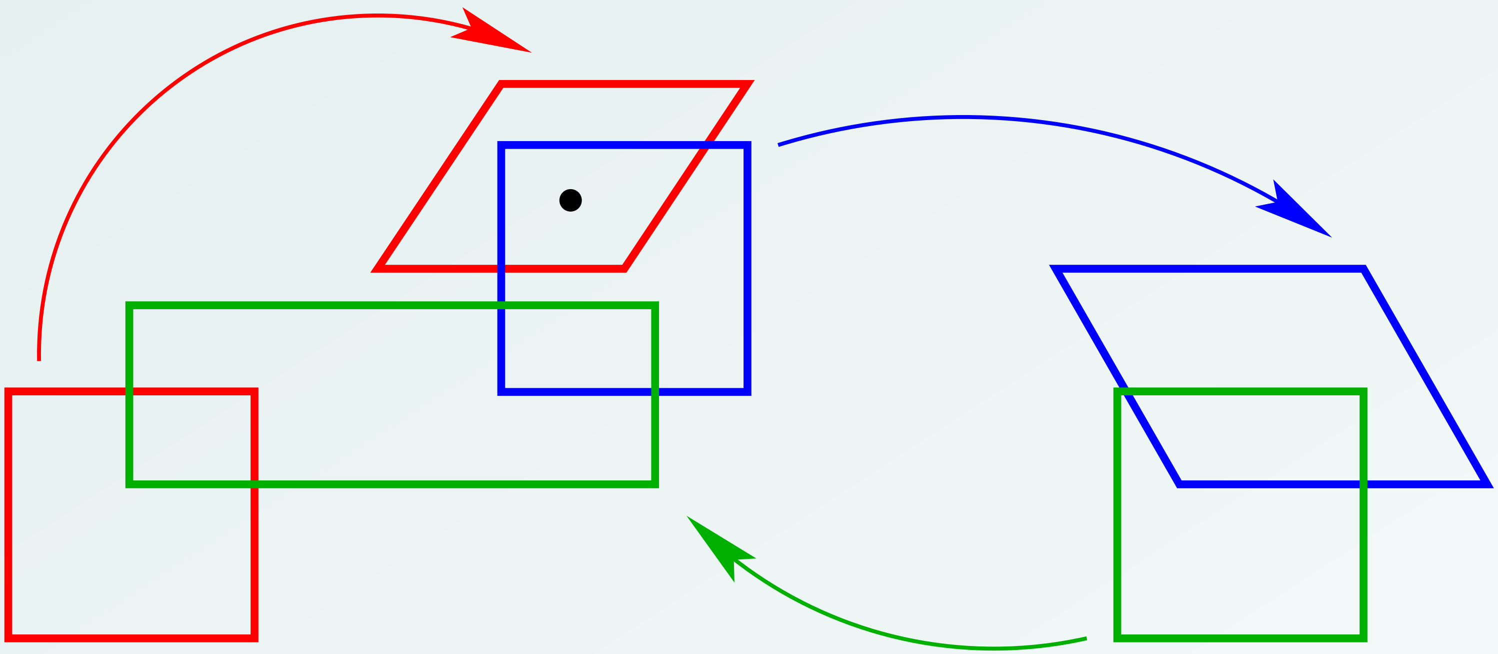

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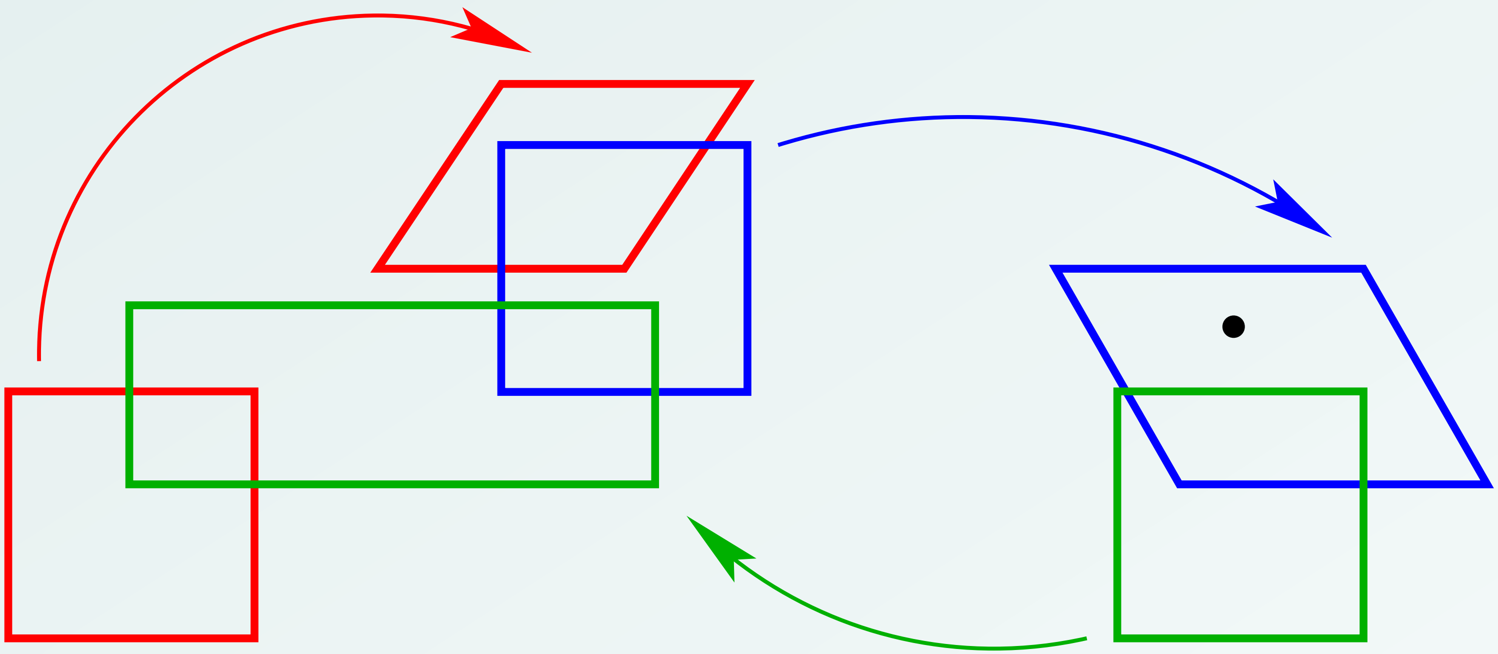
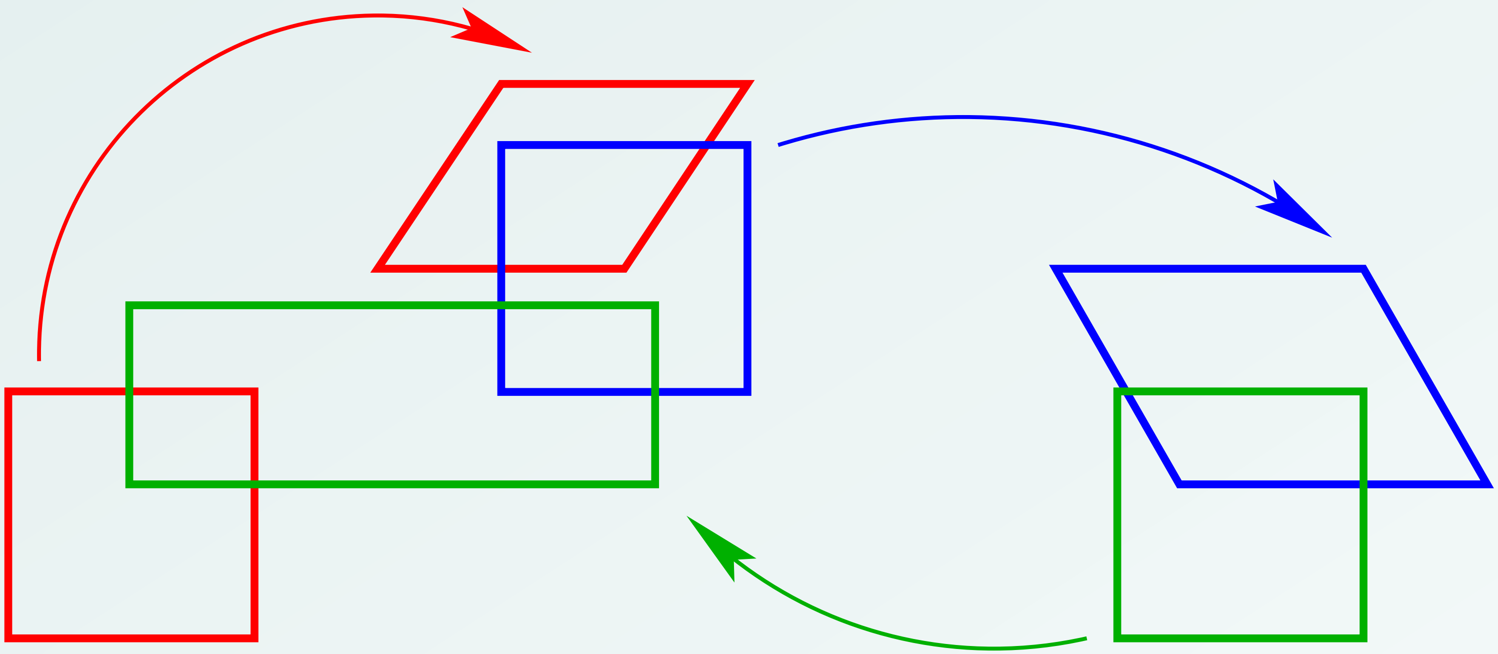


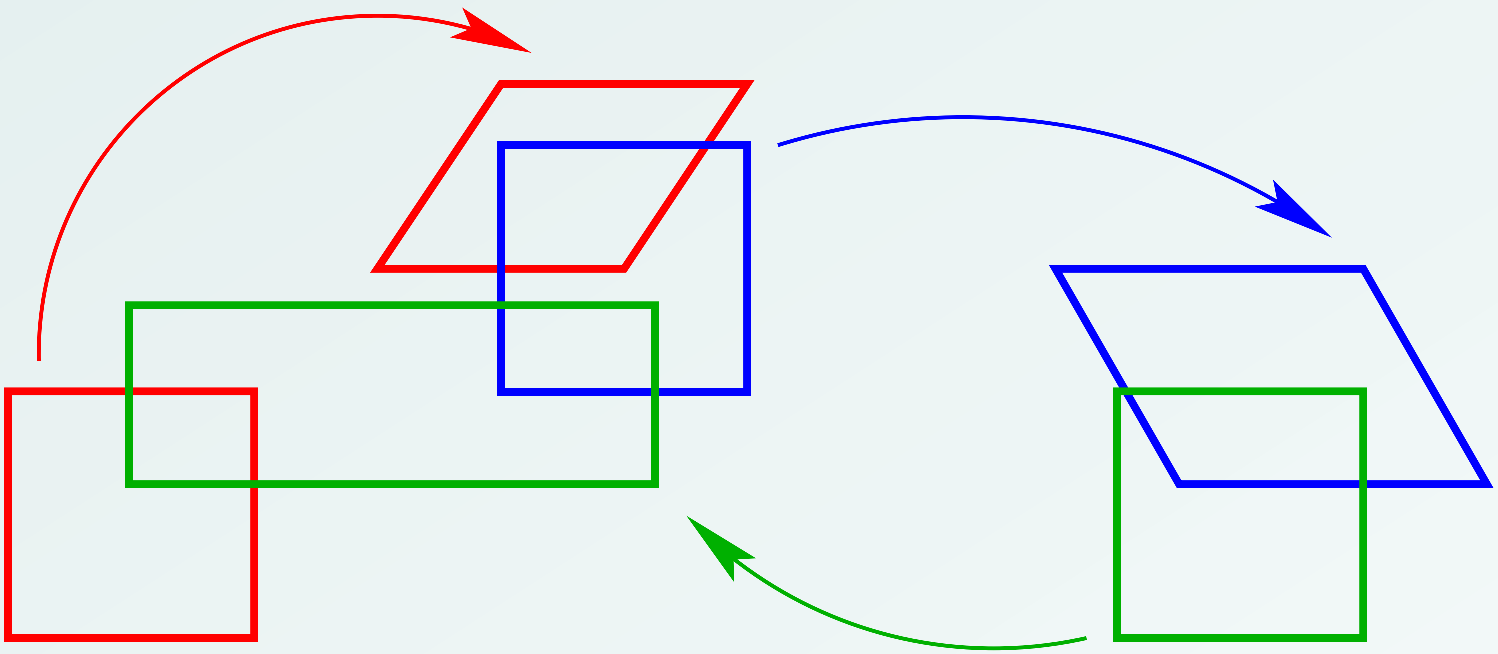
But if the point goes outside of the domain, the system **halts**.

If the iteration always halts, regardless of the starting point $\vec{x}$, the system is **mortal**. Otherwise it is **immortal**: there is an immortal point $\vec{x} \in D$ from which a non-halting orbit begins.



Immortality problem: Is a given system of affine maps immortal?

Proposition: The immortality problem is undecidable.



Immortality problem: Is a given system of affine maps immortal?

Proposition: The immortality problem is undecidable.

Follows from a standard simulation of **Turing machines** by two-dimensional piecewise affine transformations, and from:

Theorem (Hooper 1966): It is undecidable if a given Turing machine has any immortal configurations.

Next: We effectively construct Wang tiles that are forced to simulate iterations of given piecewise affine maps.

Then the undecidability of the tiling problem follows: a valid tiling exists if and only if the dynamical system has an infinite orbit (which is undecidable).

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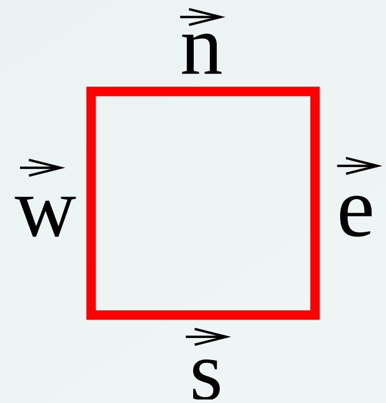
The construction is very similar to the earlier construction of 14 aperiodic tiles.

The colors in our Wang tiles are elements of $\mathbb{R}^2$.

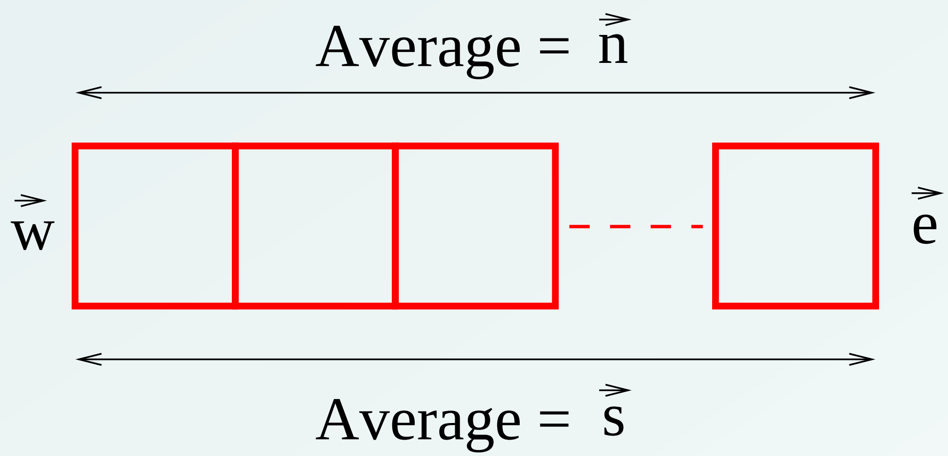
Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an affine function. We say that tile



computes function f if

$$f(\vec{n}) + \vec{w} = \vec{s} + \vec{e}.$$

Suppose we have a correctly tiled horizontal segment of length n where all tiles compute the same f.



It easily follows that

$$f(\vec{n}) + \frac{1}{n}\vec{w} = \vec{s} + \frac{1}{n}\vec{e},$$

where $\vec{n}$ and $\vec{s}$ are the averages of the top and the bottom labels.

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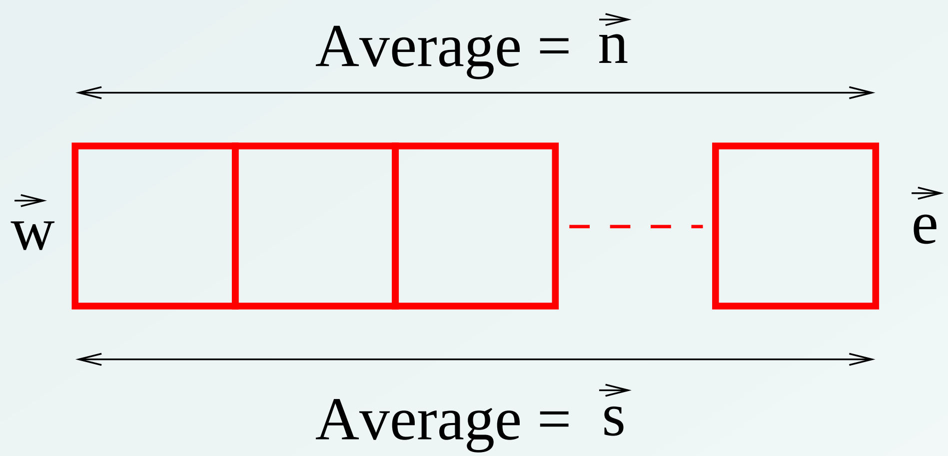

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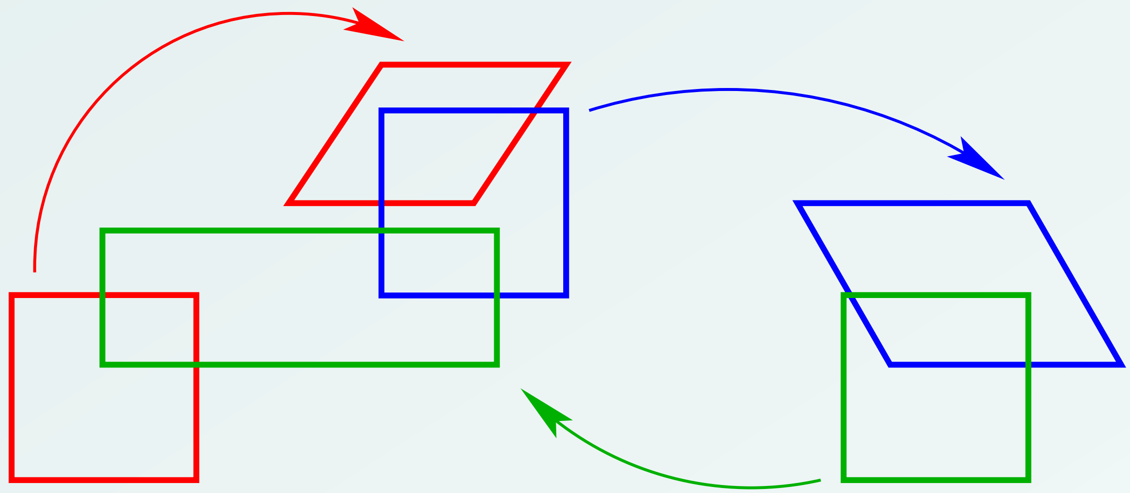
As the segment is made longer, the effect of the carry-in and carry-out labels $\vec{w}$ and $\vec{e}$ vanish.



Consider a system of affine maps f_i and unit squares U_i.

For each i we construct a set T_i of Wang tiles

- that compute function f_i, and
- whose top edge labels $\vec{n}$ are in U_i.

We also make sure that tiles of different sets T_i and T_j cannot be mixed on any horizontal row of tiles. Let

$$T = \bigcup_i T_i.$$

Claim: If such T admits a valid tiling then the system of affine maps has an immortal point.

Indeed: An immortal point is obtained as the average of the top labels on a horizontal row of the tiling. The averages on subsequent horizontal rows below are the iterates of that point under the dynamical system.

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Indeed: An immortal point is obtained as the average of the top labels on a horizontal row of the tiling. The averages on subsequent horizontal rows below are the iterates of that point under the dynamical system.

Small technicality: If the average over an infinite horizontal row does not exist then we take an accumulation point of averages of finite segments instead. . . this always exists.

We still have to detail how to choose the tiles so that also the **converse** is true: any immortal orbit of the affine maps gives a valid tiling.

The tile set corresponding to a rational affine map

$$f_i(\vec{x}) = M\vec{x} + \vec{b}$$

and its domain square U_i consists of all tiles

$$\begin{array}{ccc} & B_k(\vec{x}) & \\ \begin{array}{r} f_i(\lfloor (k-1)\vec{x} \rfloor) \\ -\lfloor (k-1) f_i(\vec{x}) \rfloor \\ +(k-1)\vec{b} \end{array} & \boxed{\phantom{\rule{3em}{3em}}} & \begin{array}{r} f_i(\lfloor k\vec{x} \rfloor) \\ -\lfloor k f_i(\vec{x}) \rfloor \\ +k\vec{b} \end{array} \\ & B_k(f_i(\vec{x})) & \end{array}$$

where $k \in \mathbb{Z}$ and $\vec{x} \in U_i$.

$$
\begin{array}{ccc}
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\end{array}
$$

where $k \in \mathbb{Z}$ and $\vec{x} \in U_i$.

(1) For fixed $\vec{x} \in U_i$ the tiles for consecutive $k \in \mathbb{Z}$ match so that a horizontal row can be formed whose top and bottom labels read the balanced representations of $\vec{x}$ and $f_i(\vec{x})$, respectively.

$$\begin{array}{ccc} & B_k(\vec{x}) & \\ \begin{array}{l} f_i(\lfloor (k-1)\vec{x}\rfloor) \\ \quad -\lfloor (k-1) f_i(\vec{x})\rfloor \\ \qquad +(k-1)\vec{b}\end{array} & \boxed{} & \begin{array}{l} f_i(\lfloor k\vec{x}\rfloor) \\ \quad -\lfloor k f_i(\vec{x})\rfloor \\ \qquad +k\vec{b}\end{array} \\ & B_k(f_i(\vec{x})) & \end{array}$$

where $k \in \mathbb{Z}$ and $\vec{x} \in U_i$.

(2) A direct calculation shows that the tile computes function f_i, that is,

$$f_i(\vec{n}) + \vec{w} = \vec{s} + \vec{e}.$$

$$B_k(\vec{x})$$

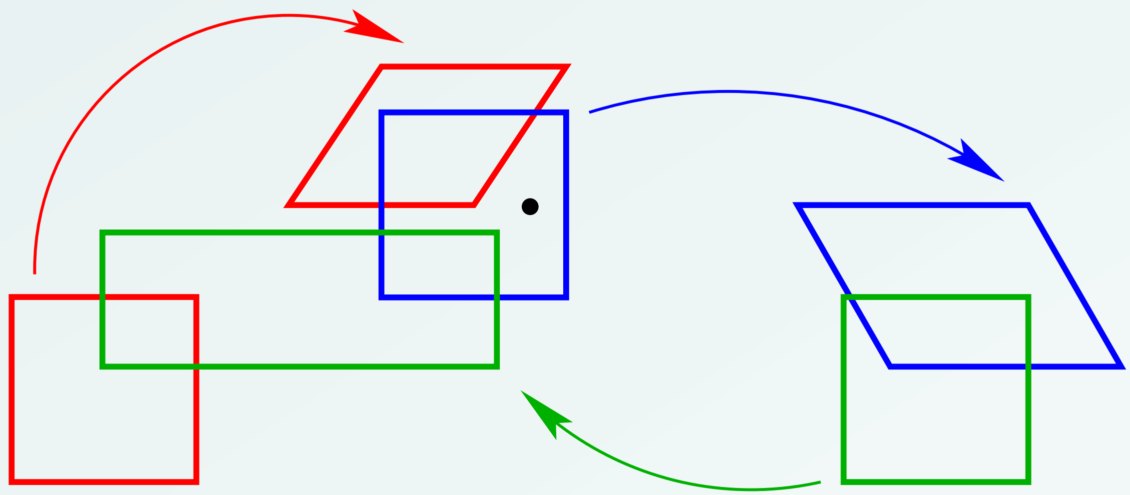
$$f_i(\lfloor (k-1)\vec{x} \rfloor) - \lfloor (k-1) f_i(\vec{x}) \rfloor + (k-1)\vec{b} \qquad\qquad f_i(\lfloor k\vec{x} \rfloor) - \lfloor k f_i(\vec{x}) \rfloor + k\vec{b}$$

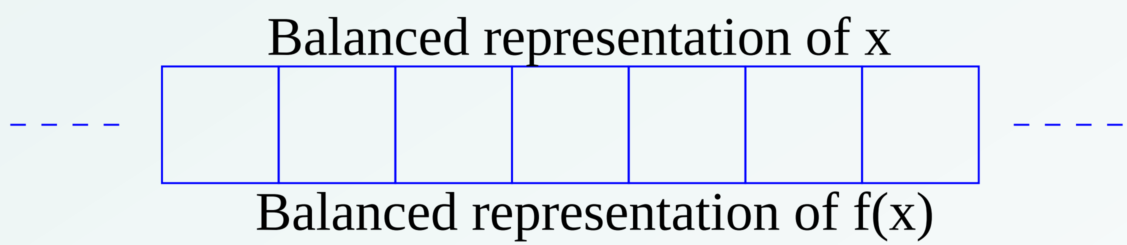
$$B_k(f_i(\vec{x}))$$

where $k \in \mathbb{Z}$ and $\vec{x} \in U_i$.

(3) Because f_i is rational, there are only finitely many such tiles (even though there are infinitely many $k \in \mathbb{Z}$ and $\vec{x} \in U_i$). The tiles can be effectively constructed.

If there is an infinite orbit then a tiling exists where the labels of the horizontal rows read the balanced representations of the points of the orbit:





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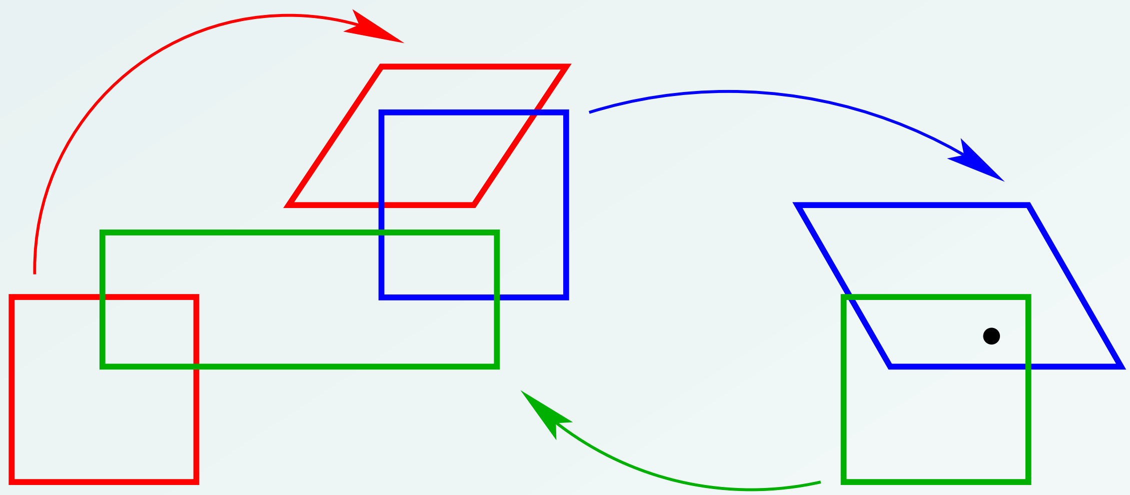

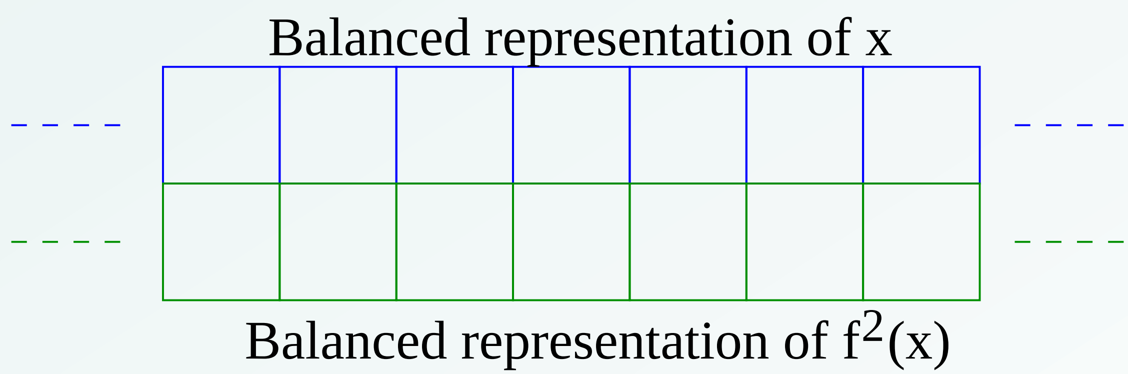

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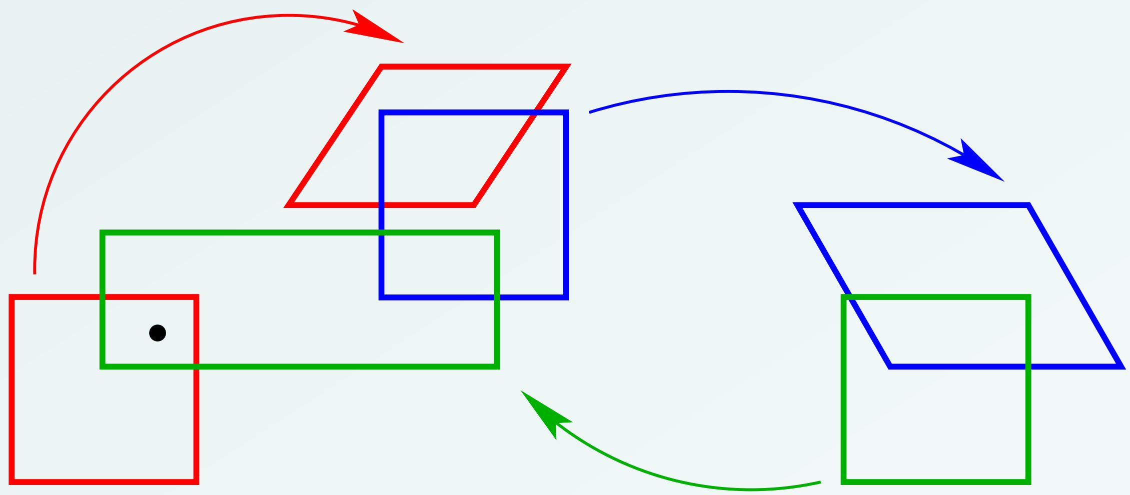

Balanced representation of x

Balanced representation of $f^3(x)$

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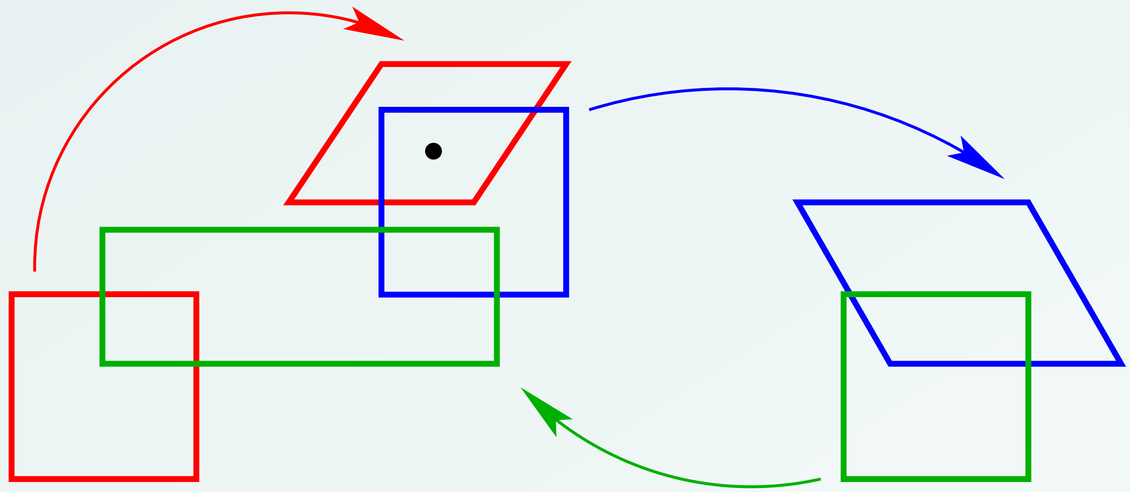

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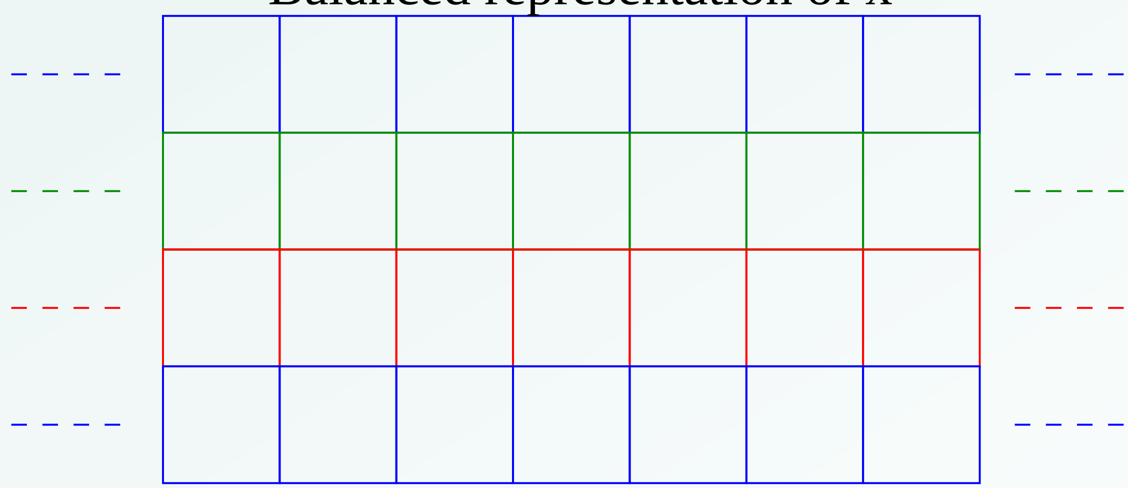
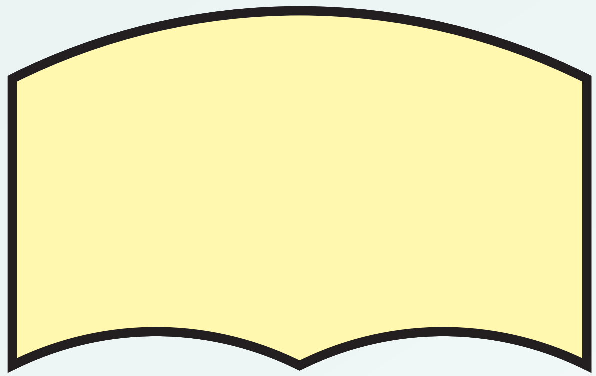


Balanced representation of $f^4(x)$

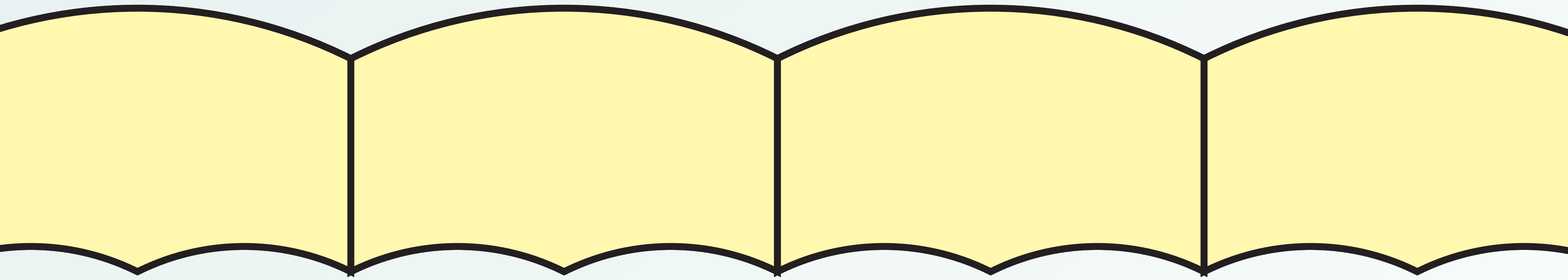
Conclusion: the tile set admits a tiling of the plane if and only if the system of affine maps is immortal. Undecidability of the tiling problem follows from the undecidability of the immortality problem. $\square$

The hyperbolic plane

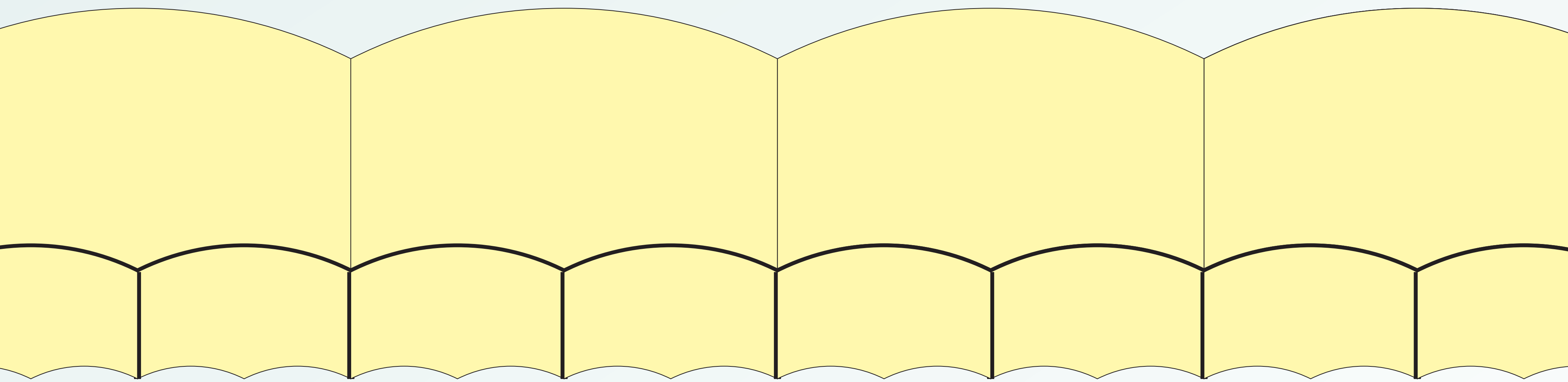
The technique works well also in the hyperbolic plane.



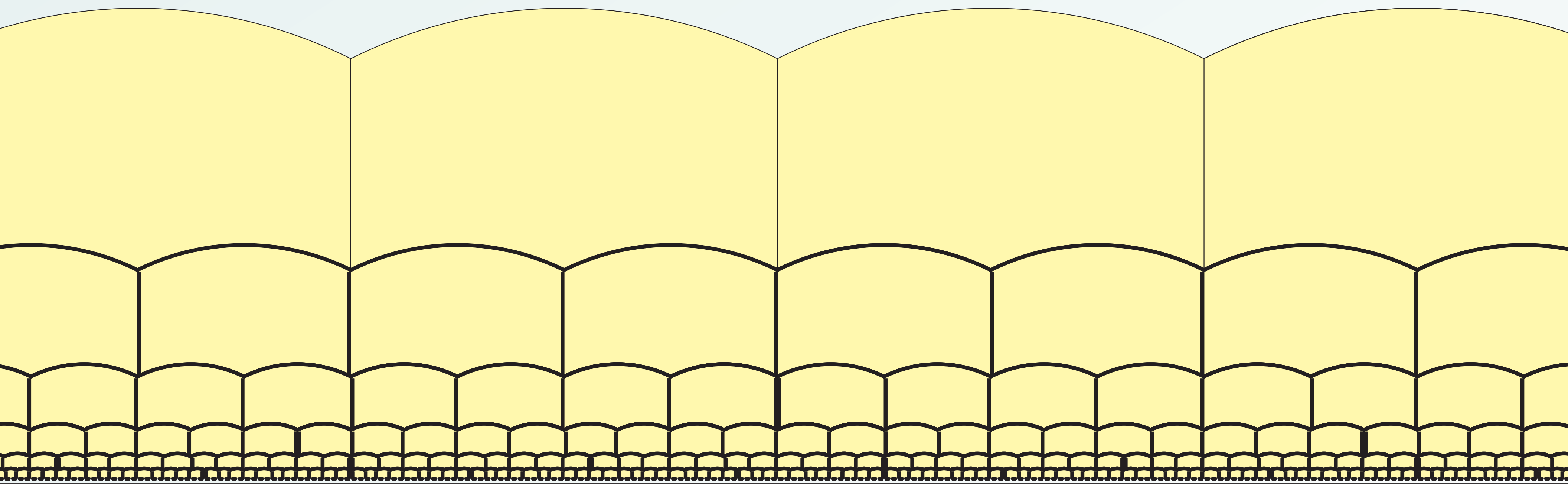
The role of the Euclidean Wang square tile will be played by a hyperbolic pentagon.



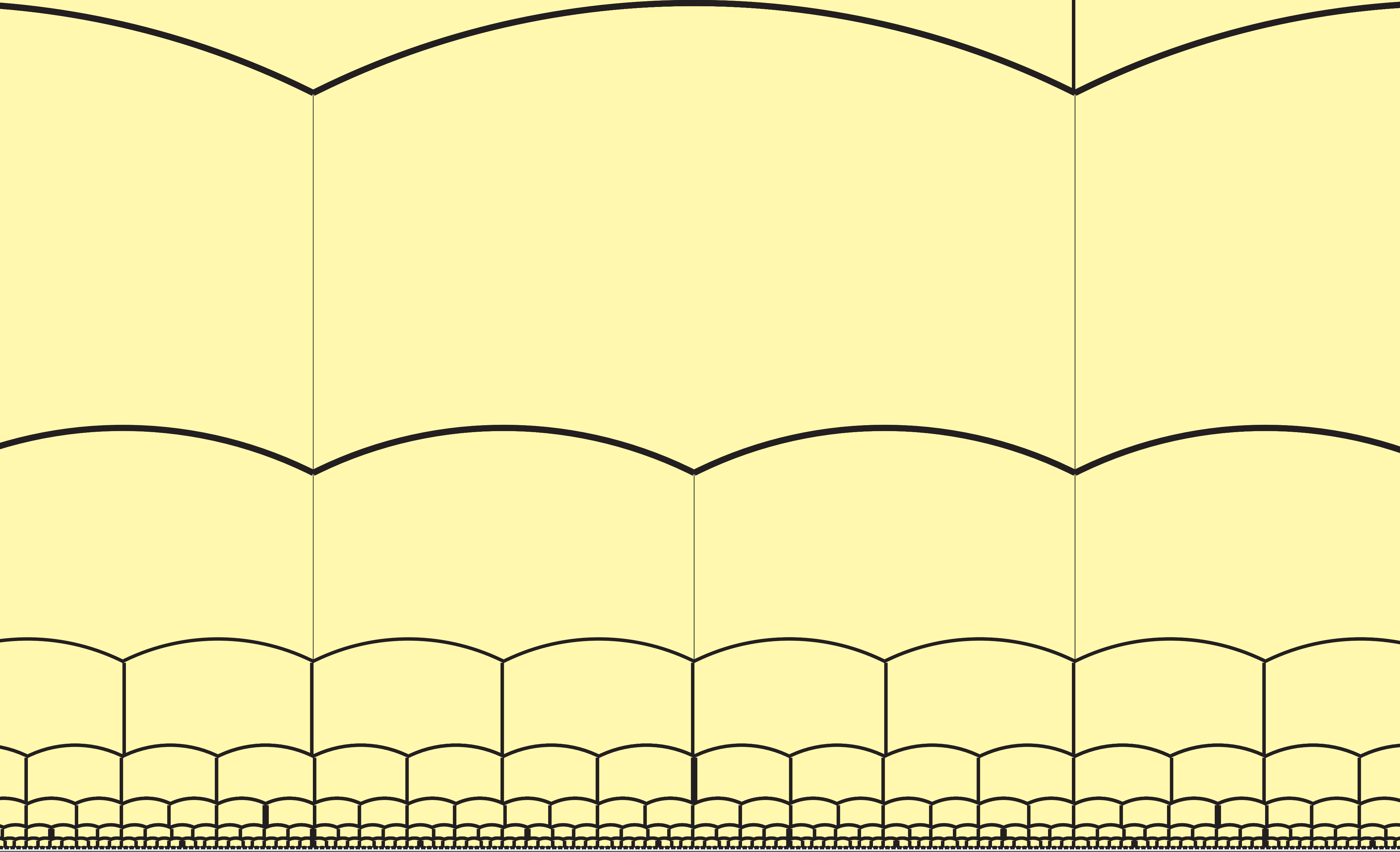
The pentagons can tile a ”horizontal row”.



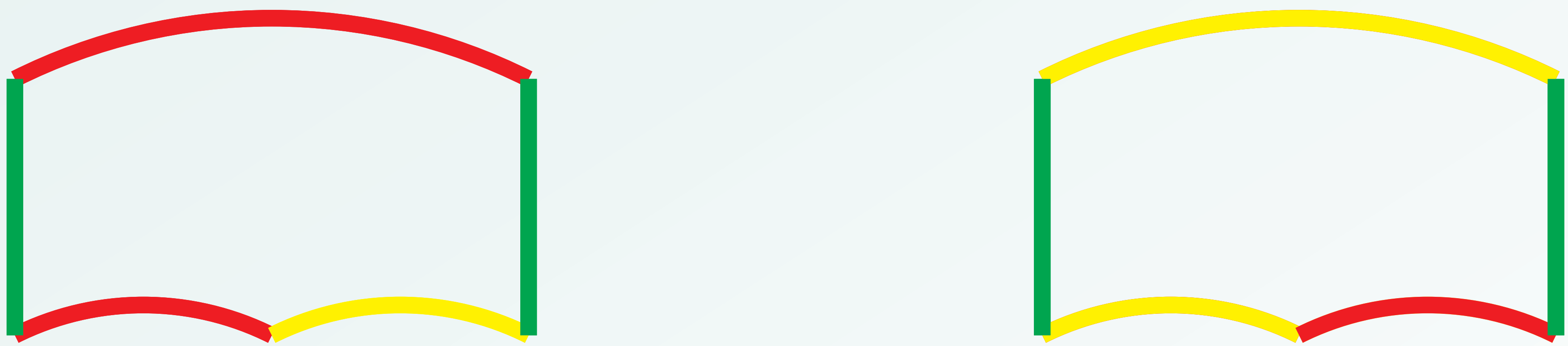
”Beneath” each pentagon fits two identical pentagons.



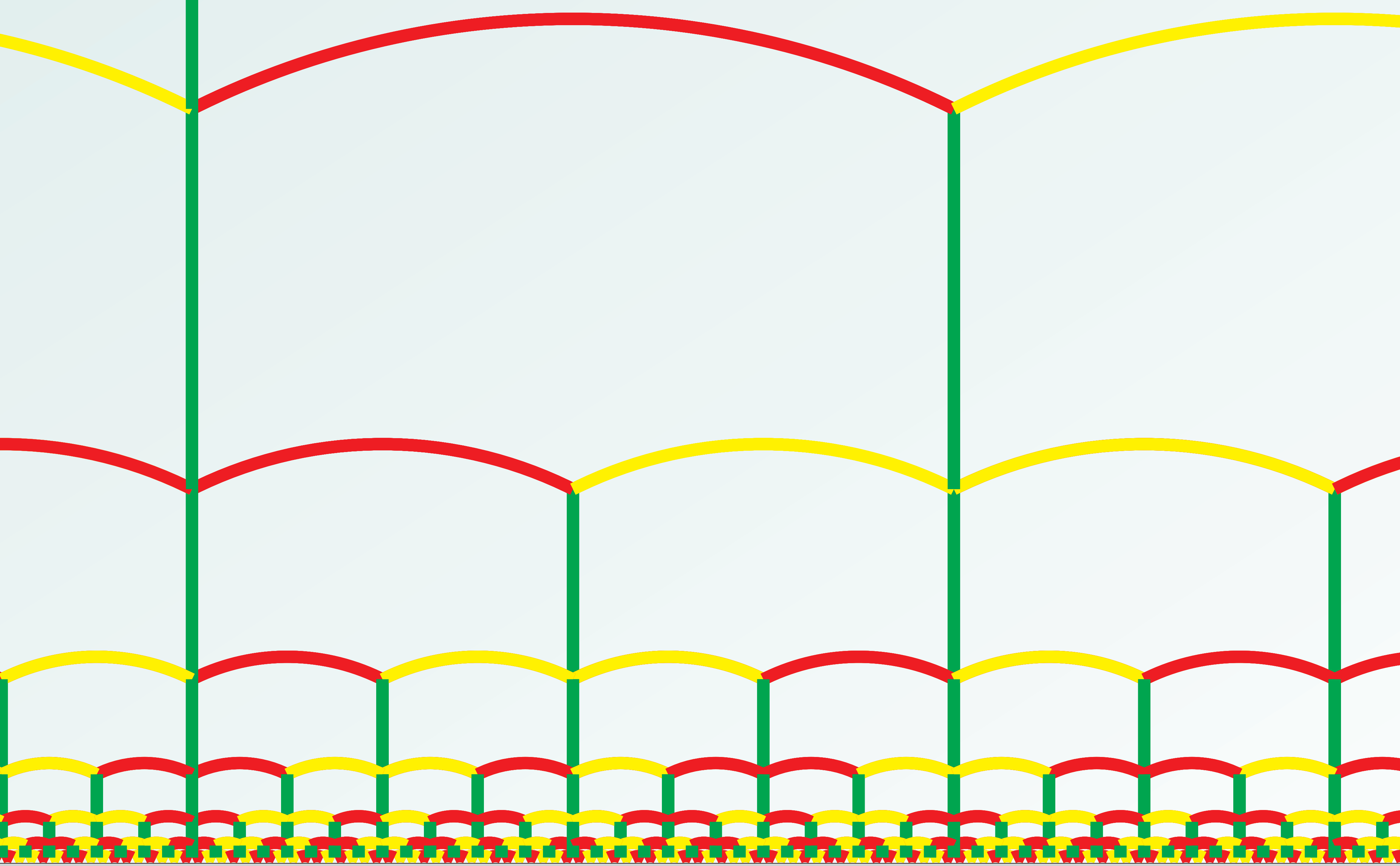
Infinitely many ”horizontal rows” fill the lower part of the half plane.



Similarily the upper part can be filled. We see that the pentagons tile the hyperbolic plane (in an uncountable number of different ways, in fact.)



On the hyperbolic plane Wang tiles are pentagons with colored edges. Pentagons may be placed adjacent if the edge colors match.

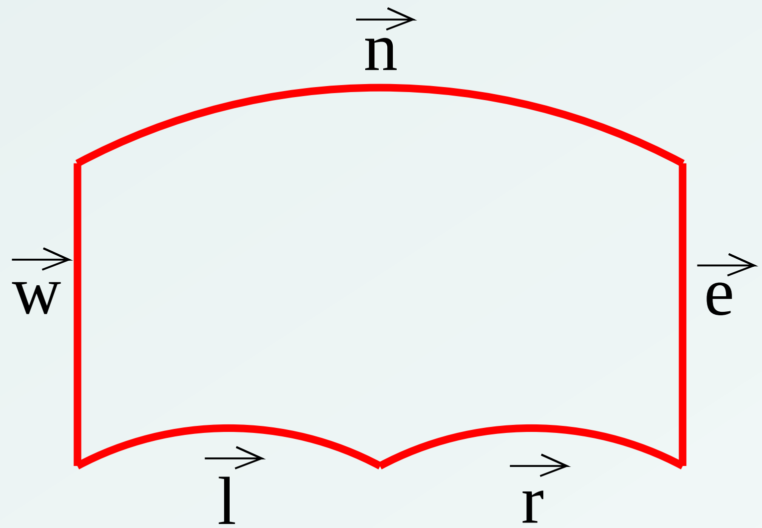


A given set of pentagons tiles the hyperbolic plane if a tiling exists where the color constraint is everywhere satisfied.

The hyperbolic tiling problem asks whether a given finite collection of colored pentagons admits a valid tiling.

Theorem. The tiling problem of the hyperbolic plane is undecidable.

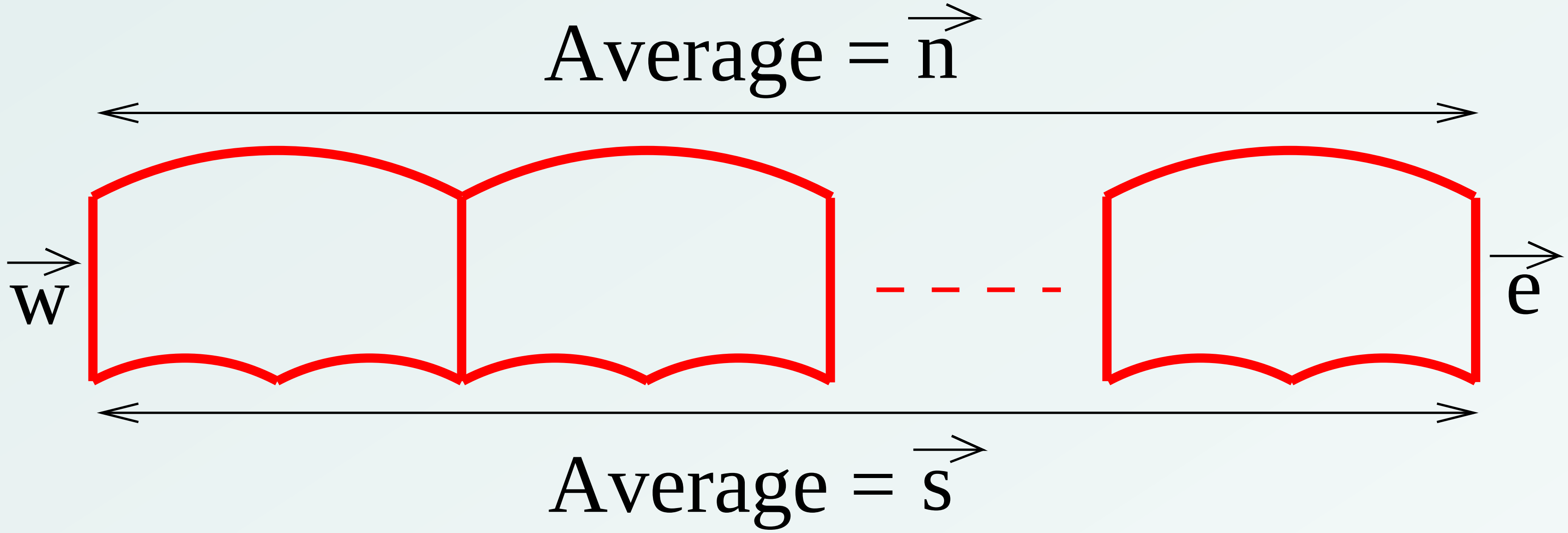
We say that pentagon



computes the affine transformation $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ if

$$f(\vec{n}) + \vec{w} = \frac{\vec{l} + \vec{r}}{2} + \vec{e}.$$

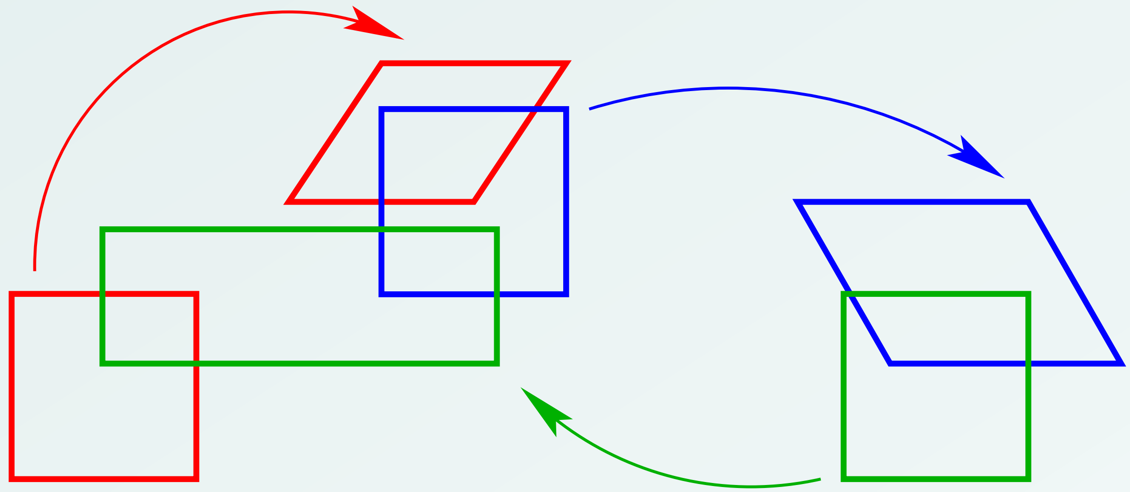
(Difference to Euclidean Wang tiles: The "output" is now divided between $\vec{l}$ and $\vec{r}$.)



In a horizontal segment of length n where all tiles compute the same f holds

$$f(\vec{n}) + \frac{1}{n}\vec{w} = \vec{s} + \frac{1}{n}\vec{e},$$

where $\vec{n}$ and $\vec{s}$ are the averages of the top and the bottom labels.

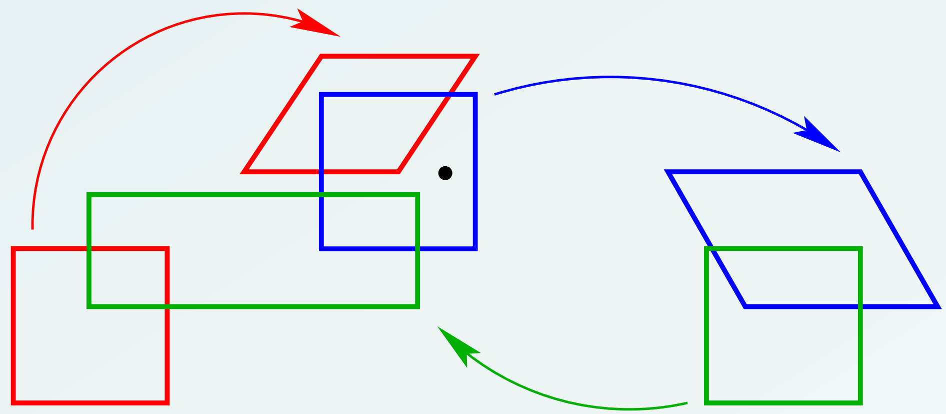


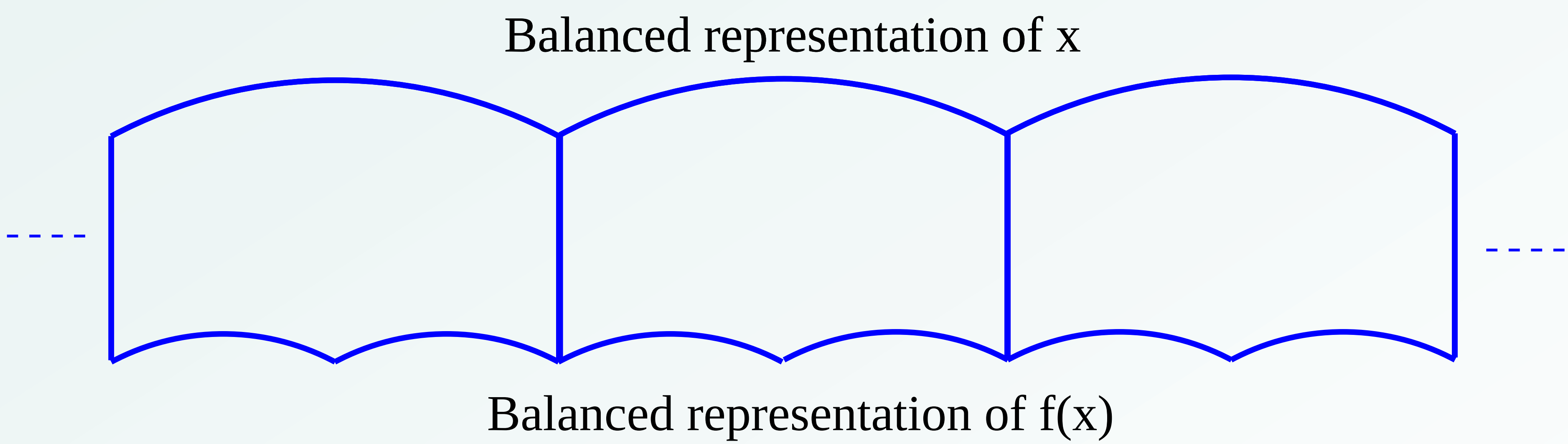
For a given system of affine maps f_i and unit squares U_i we construct for each i a set T_i of pentagons

- that compute function f_i, and
- whose top edge labels $\vec{n}$ are in U_i.

It follows, exactly as in the Euclidean case, that valid tilings correspond to iterations of the piecewise affine maps.

The tiles constructed admit a valid tiling iff the system of affine maps has an immortal point:





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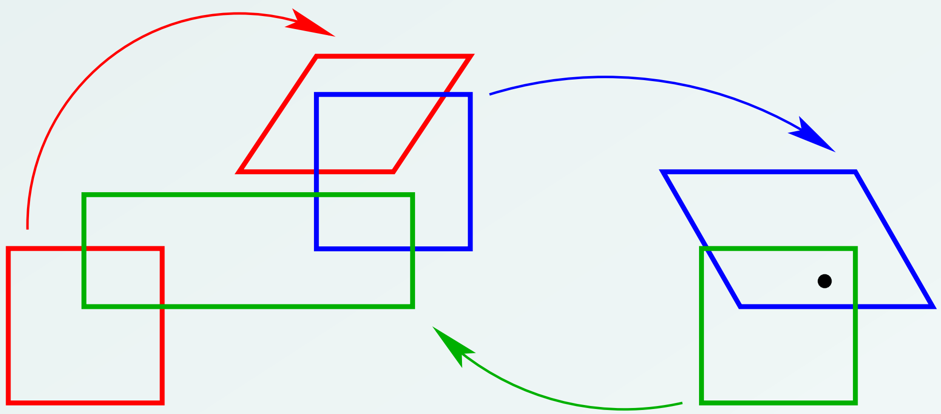

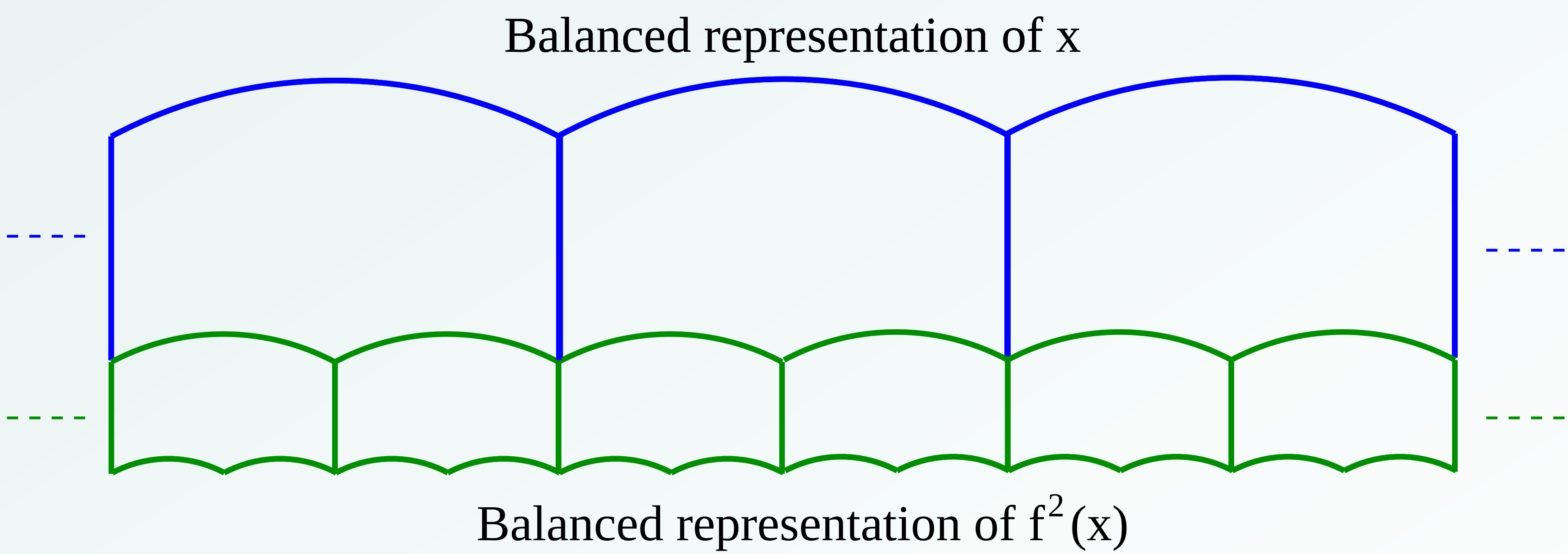

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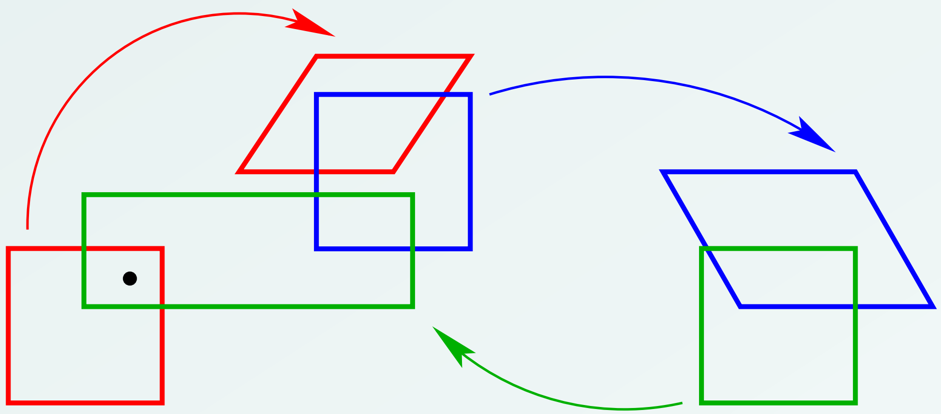
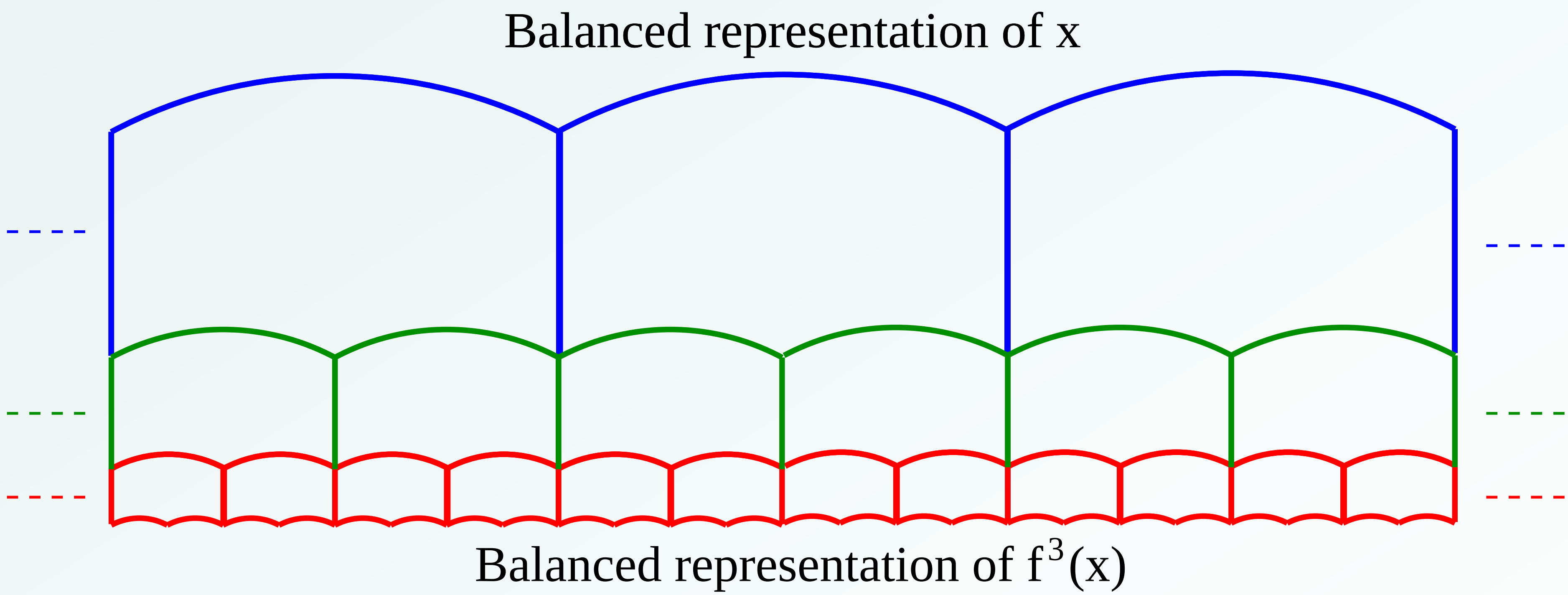
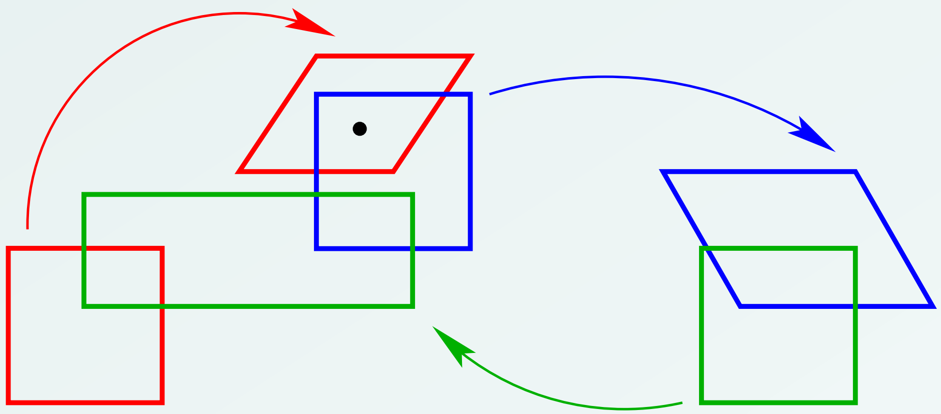
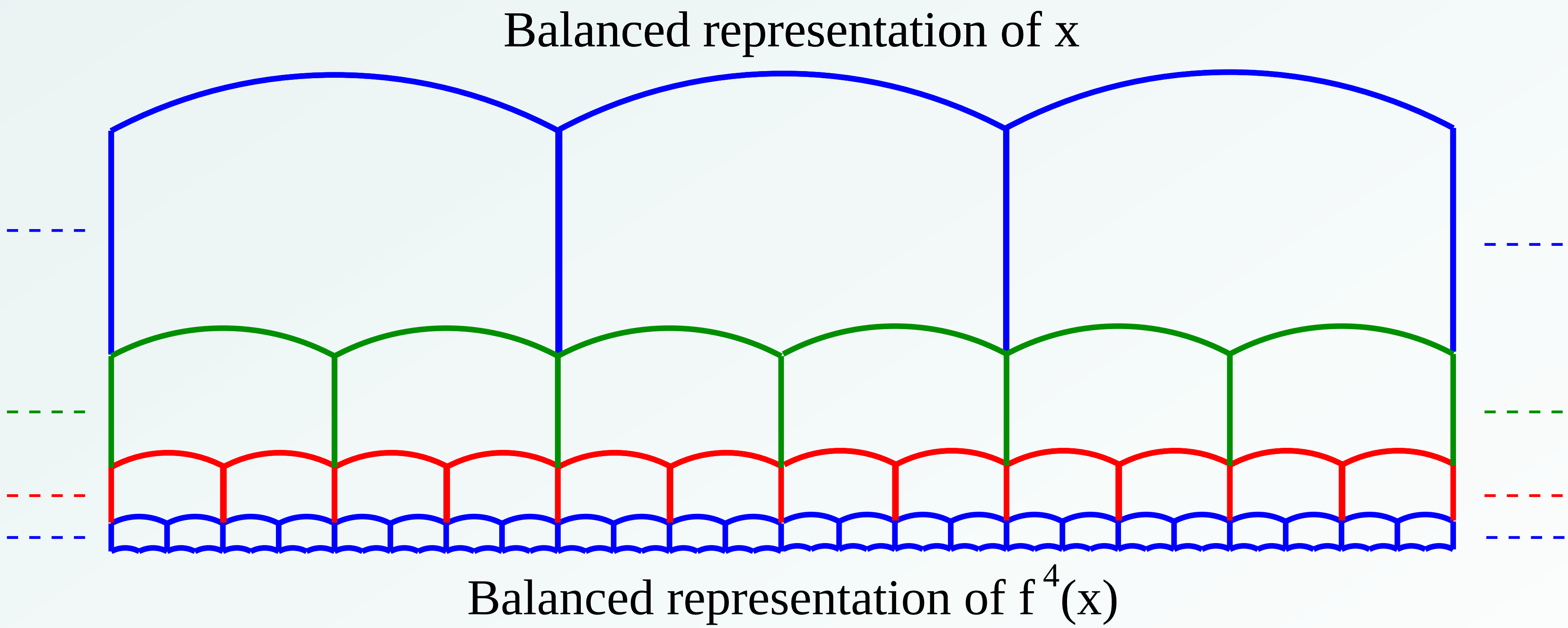




The tiles constructed admit a valid tiling iff the system of affine maps has an immortal point:





Conclusion

Sturmian representations of real numbers admit concise simulations of piecewise affine maps on 2D tilings.

$\Longrightarrow$ small aperiodic sets of Wang tiles

$\Longrightarrow$ simple undecidability proof of the tiling problem

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On which groups is the tiling problem decidable ?

- Decidable on **virtually free** groups.
- Undecidable on **Baumslag-Solitar** groups (JK, N.Aubrun).

Cellular Automata (CA): Introduction

Cellular automata are an old model of computation. They are investigated

- **in physics** as discrete models of physical systems,
- **in computer science** as models of massively parallel computation under the realistic constraints of locality and uniformity,
- **in mathematics** as endomorphisms of the full shift in the context of symbolic dynamics.

Cellular automata possess several fundamental properties of the physical world: they are

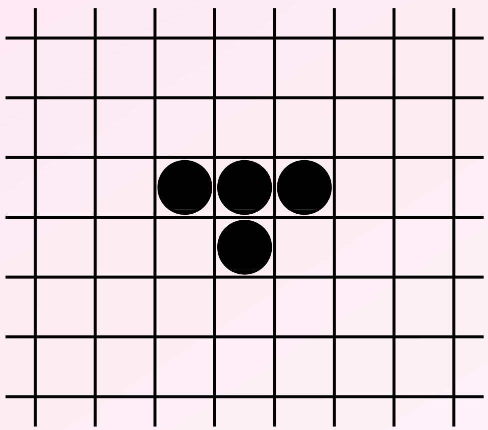
- **massively parallel**,
- **homogeneous** in time and space,
- all interactions are **local**,
- **time reversibility** and **conservation laws** can be obtained by choosing the local update rule properly.

Example: the **Game-of-Life** by John Conway.

- Infinite checker-board whose squares (=cells) are colored black (=**alive**) or white (=**dead**).
- At each discrete time step each cell counts the number of living cells surrounding it, and based on this number determines its new state.
- All cells change their state simultaneously.

The local update rule asks each cell to check the present states of the eight surrounding cells.

- If the cell is **alive** then it stays alive (survives) iff it has two or three live neighbors. Otherwise it dies of loneliness or overcrowding.
- If the cell is **dead** then it becomes alive iff it has exactly three living neighbors.



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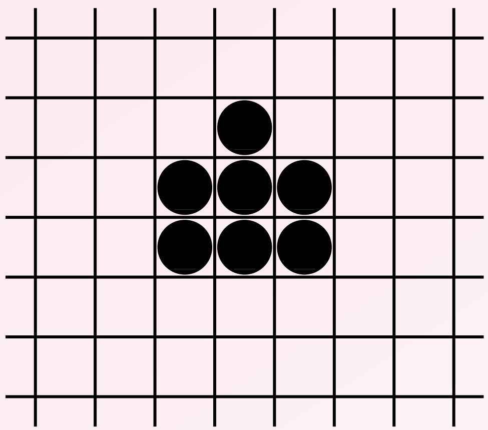

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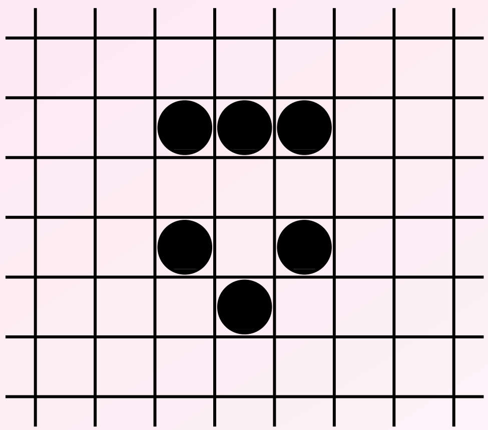

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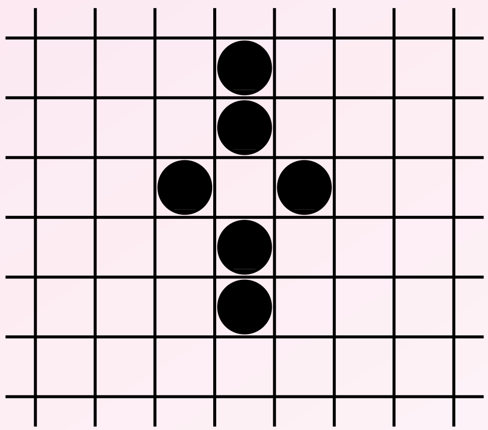
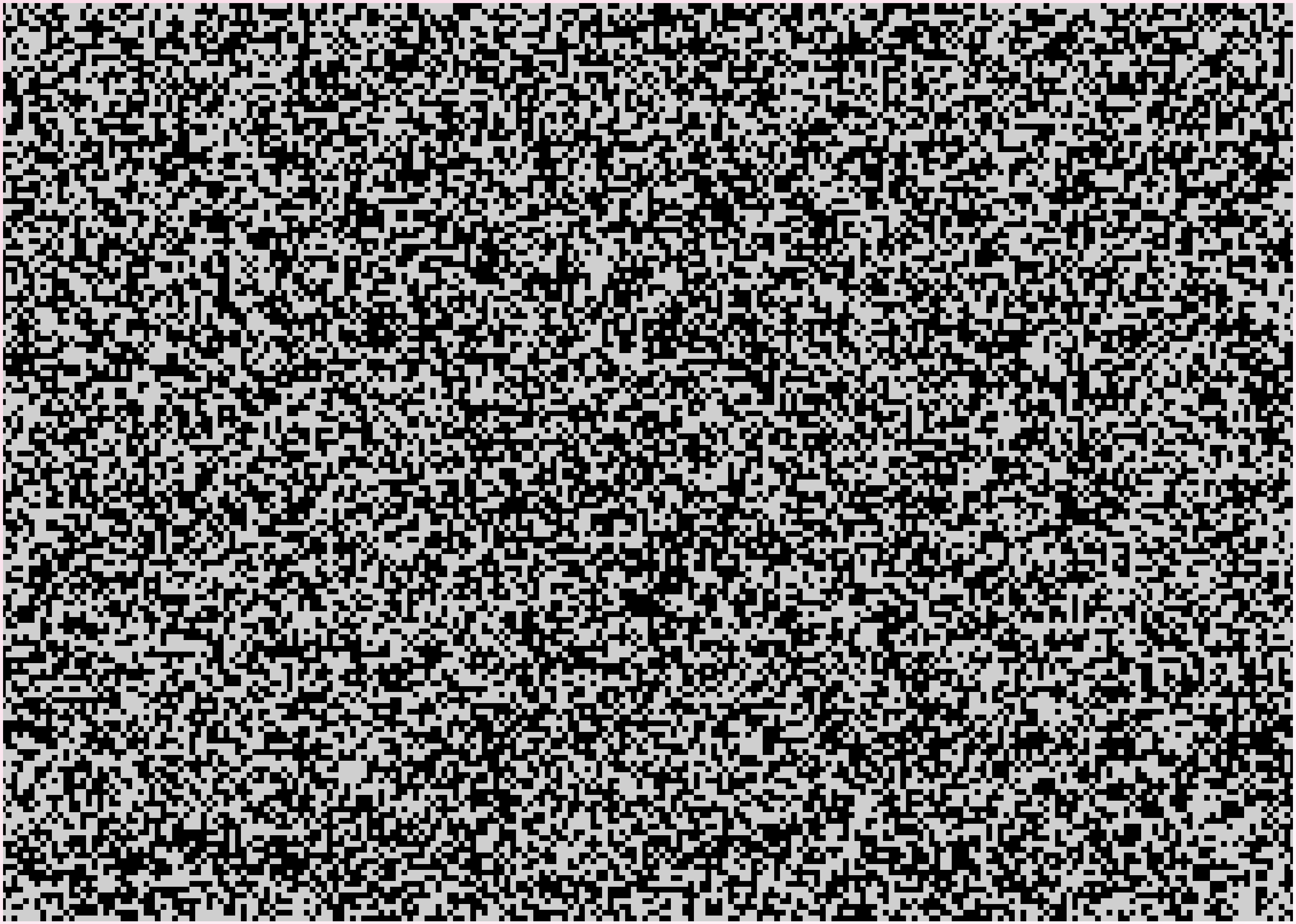
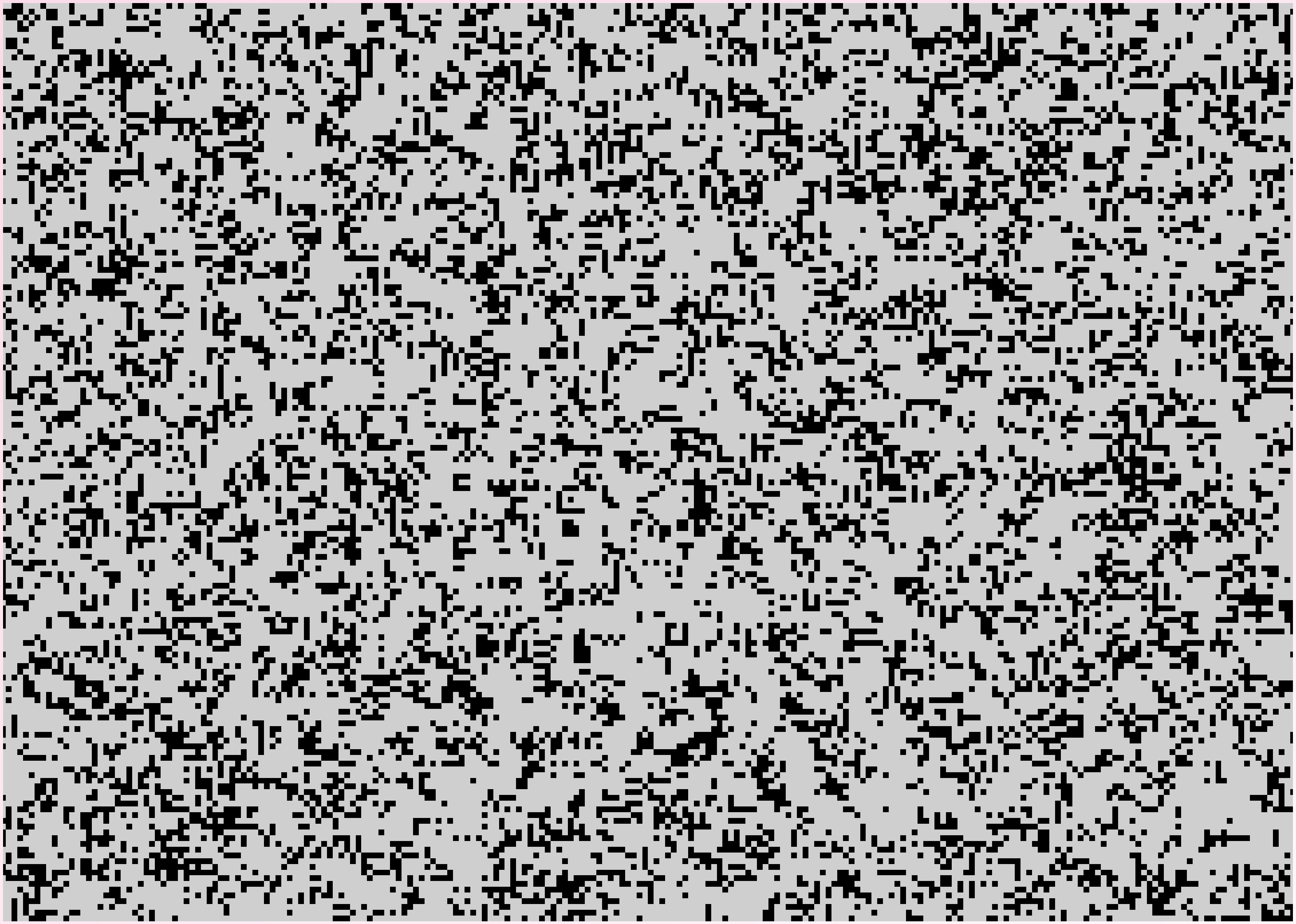


A typical snapshot of a time evolution in Game-of-Life:



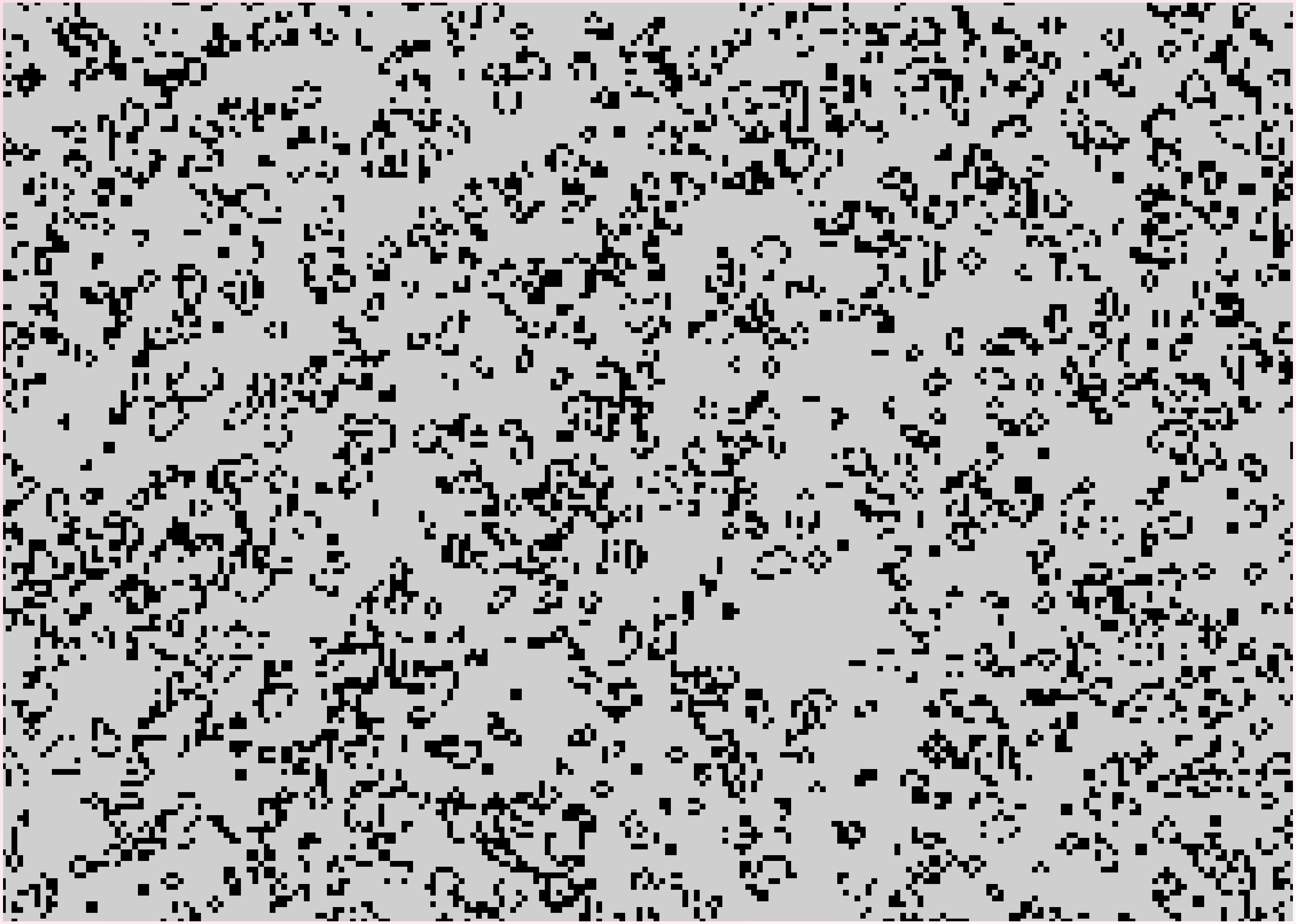
Initial uniformly random configuration.

A typical snapshot of a time evolution in Game-of-Life:



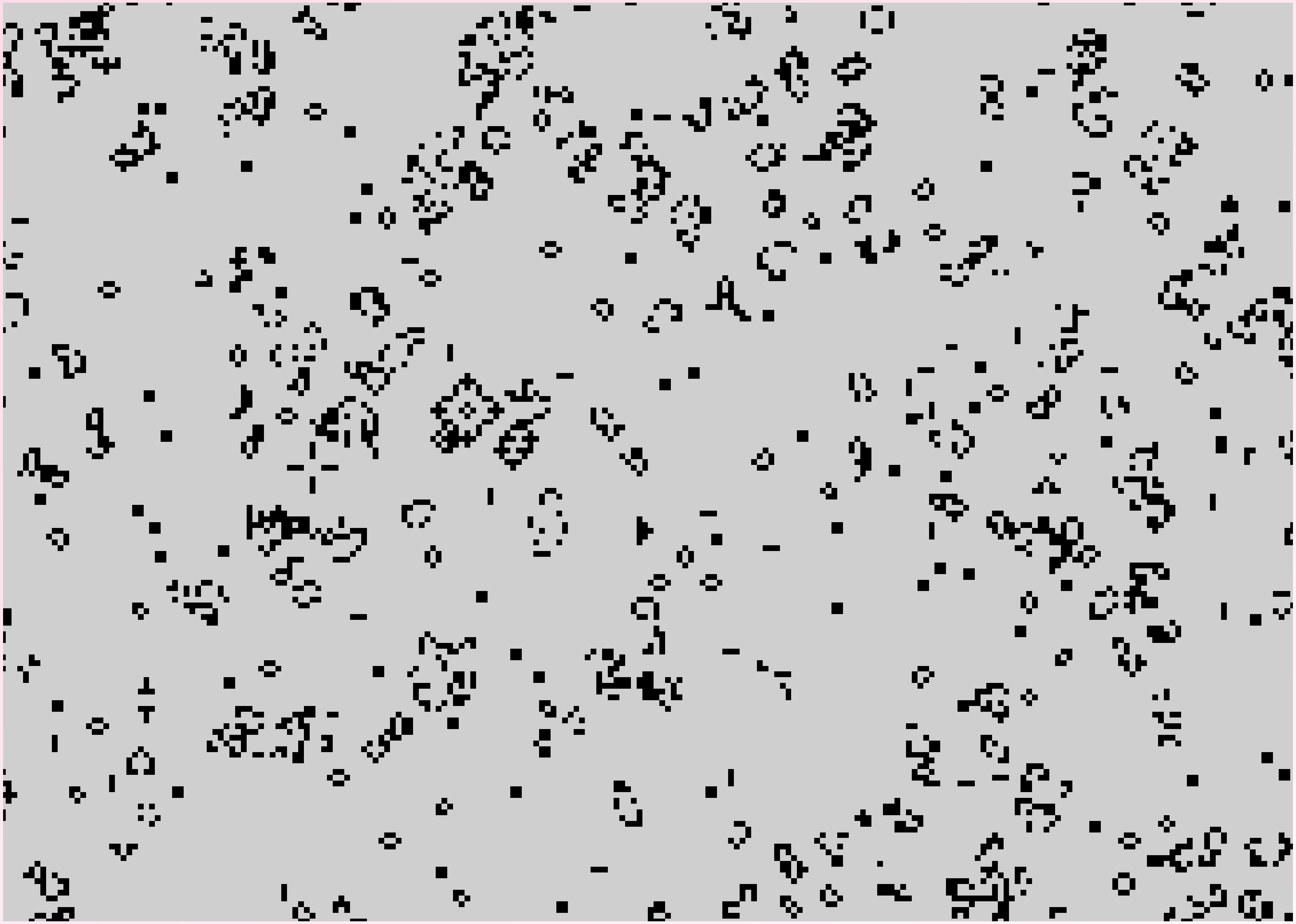
The next generation after all cells applied the update rule.

A typical snapshot of a time evolution in Game-of-Life:



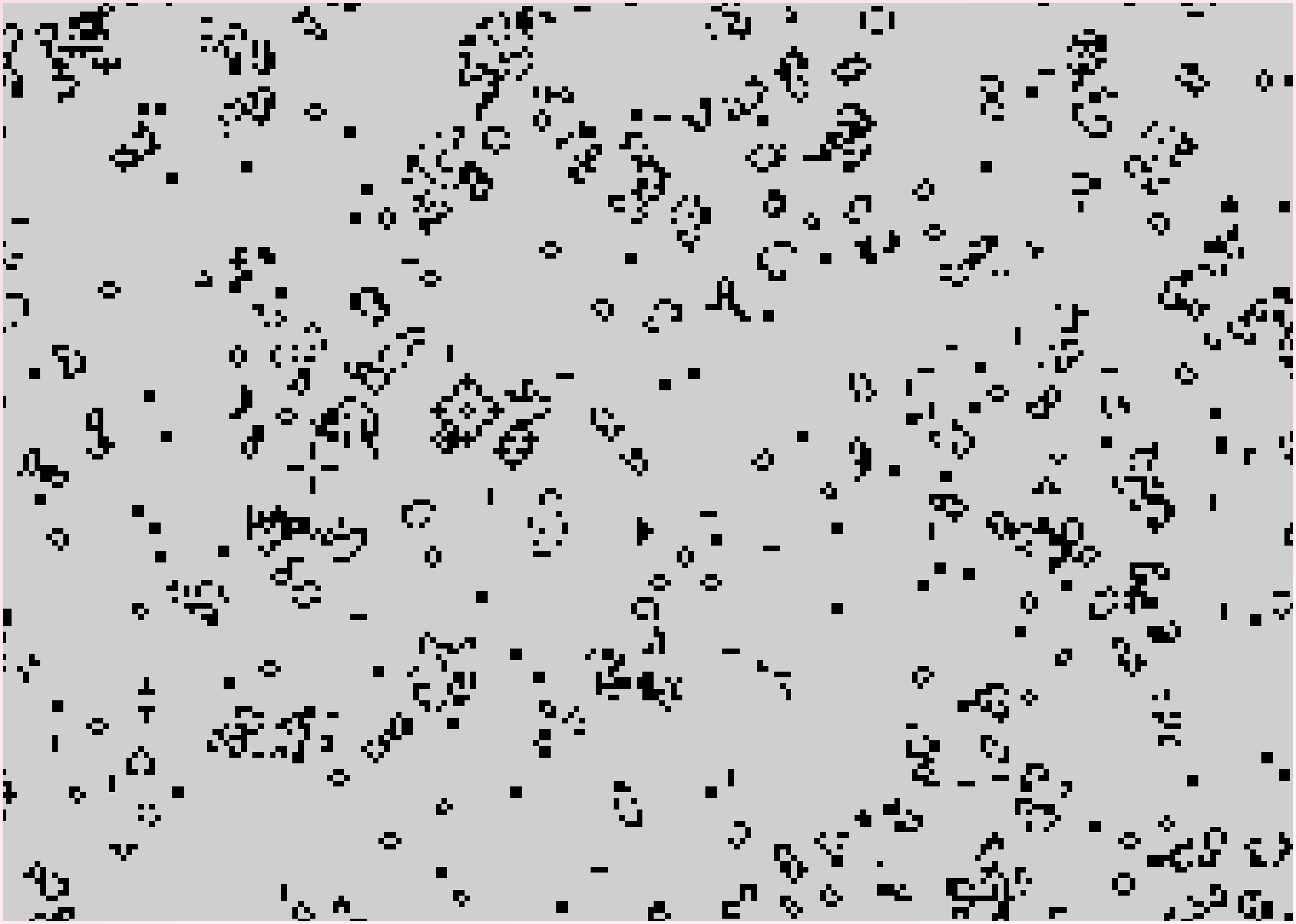
Generation 10

A typical snapshot of a time evolution in Game-of-Life:



Generation 100

A typical snapshot of a time evolution in Game-of-Life:



GOL is a computationally universal two-dimensional CA.

Another famous universal CA: **rule 110** by S.Wolfram.

A one-dimensional CA with binary state set $\{0, 1\}$, i.e. a two-way infinite sequence of 0's and 1's.

Each cell is updated based on its old state and the states of its left and right neighbors as follows:

$$\begin{array}{ccc} 111 & \longrightarrow & 0 \\ 110 & \longrightarrow & 1 \\ 101 & \longrightarrow & 1 \\ 100 & \longrightarrow & 0 \\ 011 & \longrightarrow & 1 \\ 010 & \longrightarrow & 1 \\ 001 & \longrightarrow & 1 \\ 000 & \longrightarrow & 0 \end{array}$$

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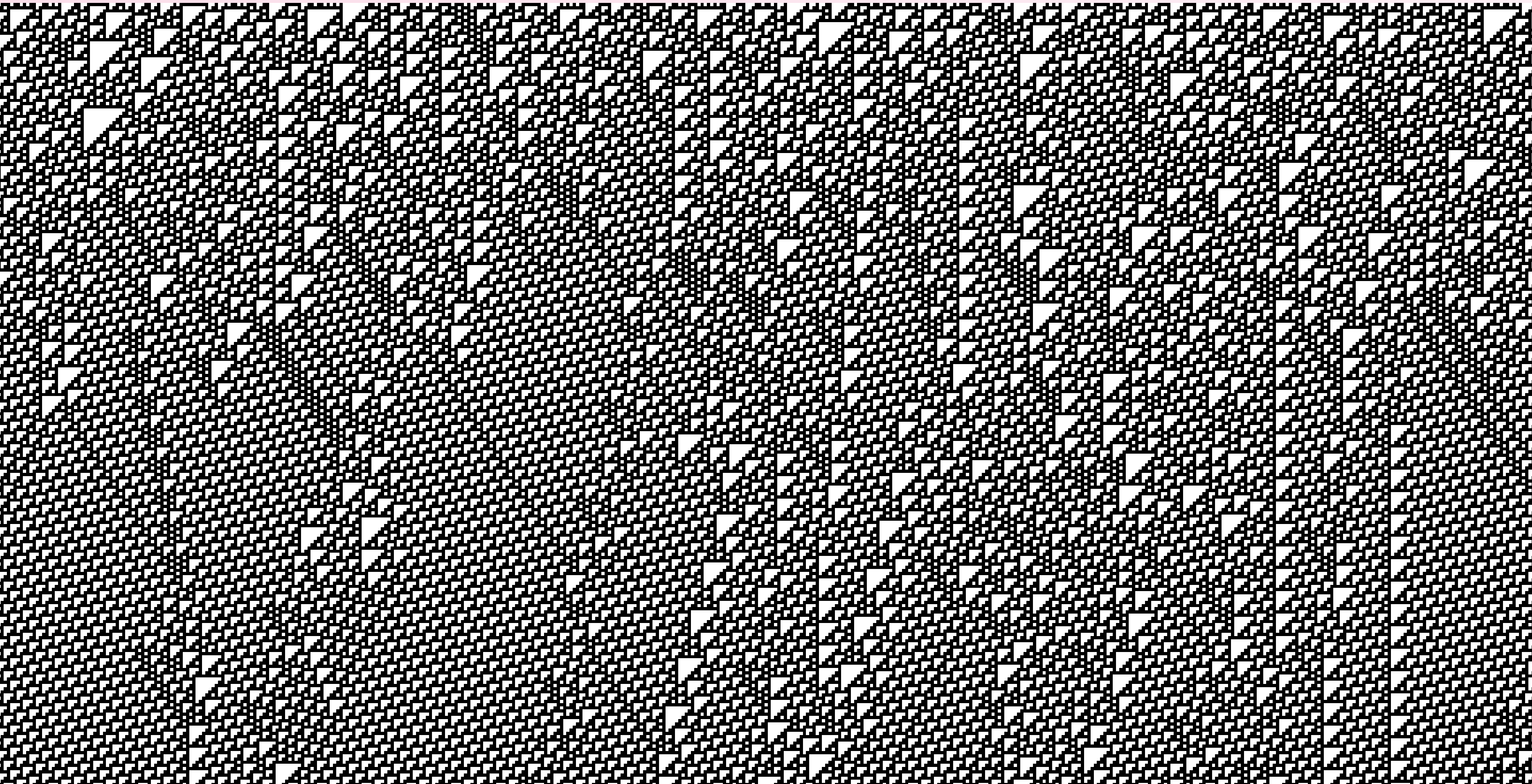
110 is the **Wolfram number** of this CA rule.

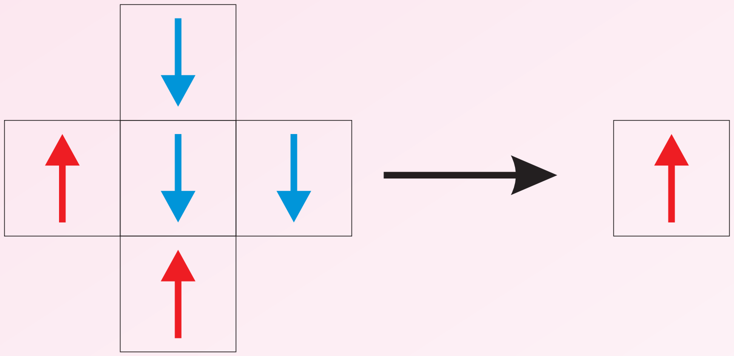
Space-time diagram is a pictorial representation of a time evolution in one-dimensional CA, where space and time are represented by the horizontal and vertical direction:



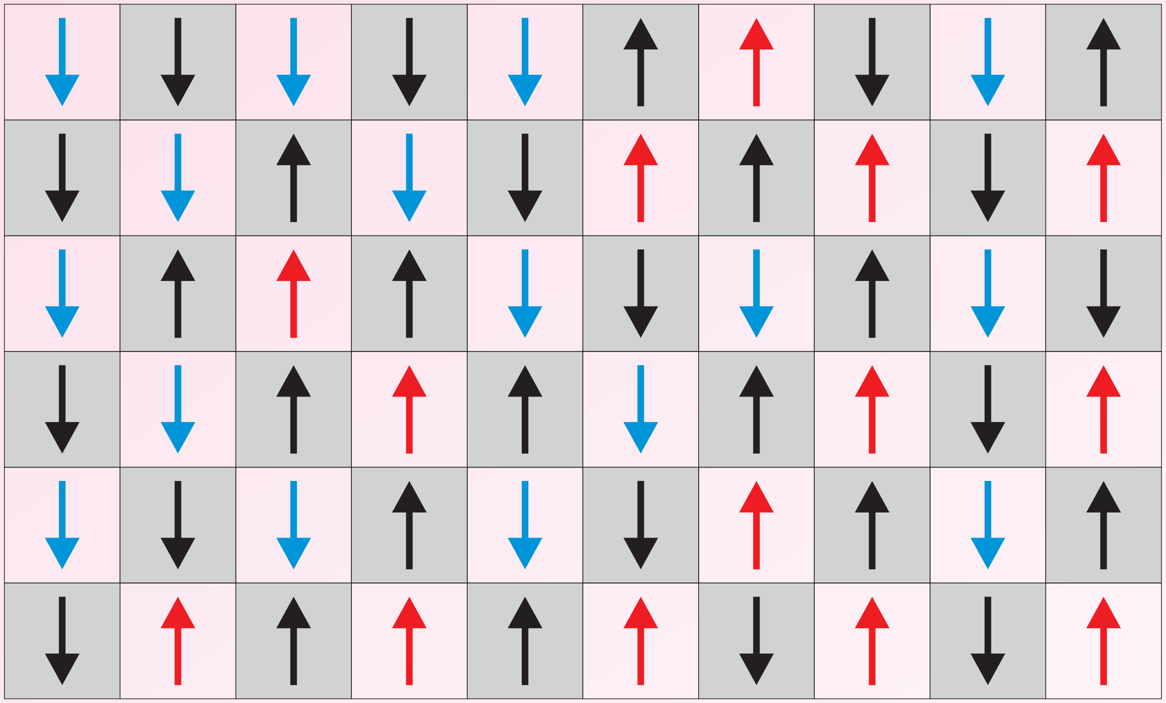
Game-of-Life and Rule 110 are **irreversible**: Configurations may have several pre-images.

Two-dimensional **Q2R** Ising model by G.Vichniac (1984) is an example of a time-reversible cellular automaton.

Each cell has a spin that is directed either up or down. The direction of a spin is swapped if and only if among the four immediate neighbors there are exactly two cells with spin up and two cells with spin down:



The twist that makes the Q2R rule reversible: Color the space as a checker-board. On even time steps only update the spins of the white cells and on odd time steps update the spins of the black cells.



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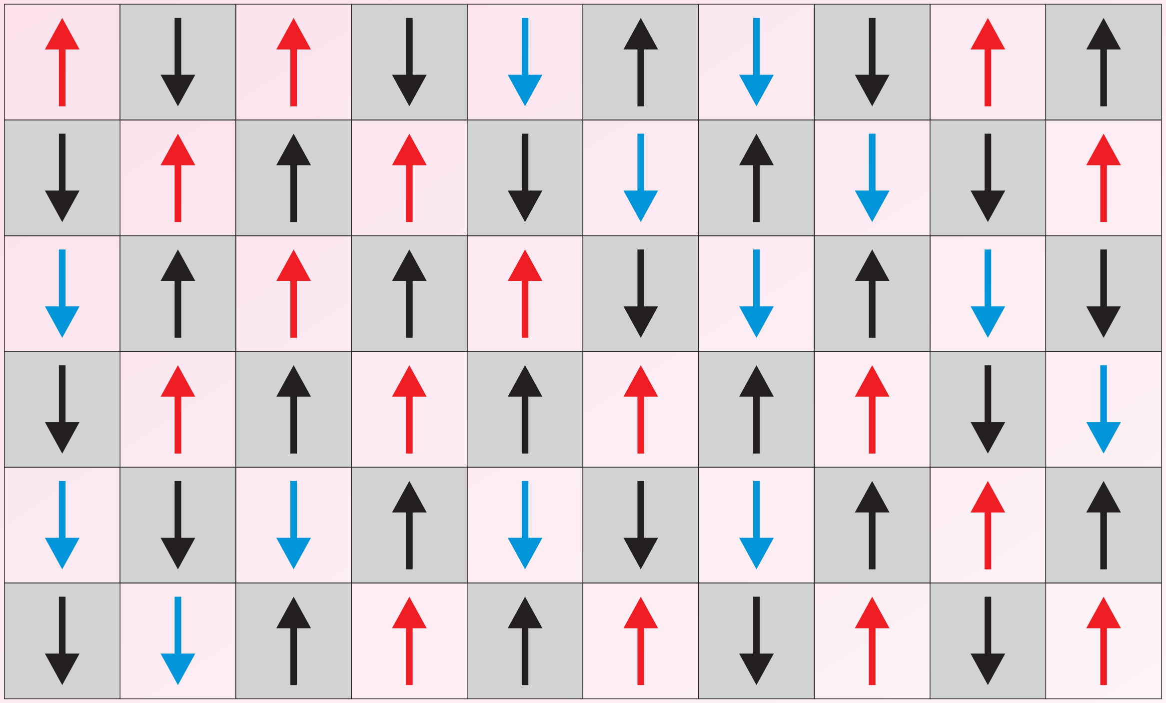

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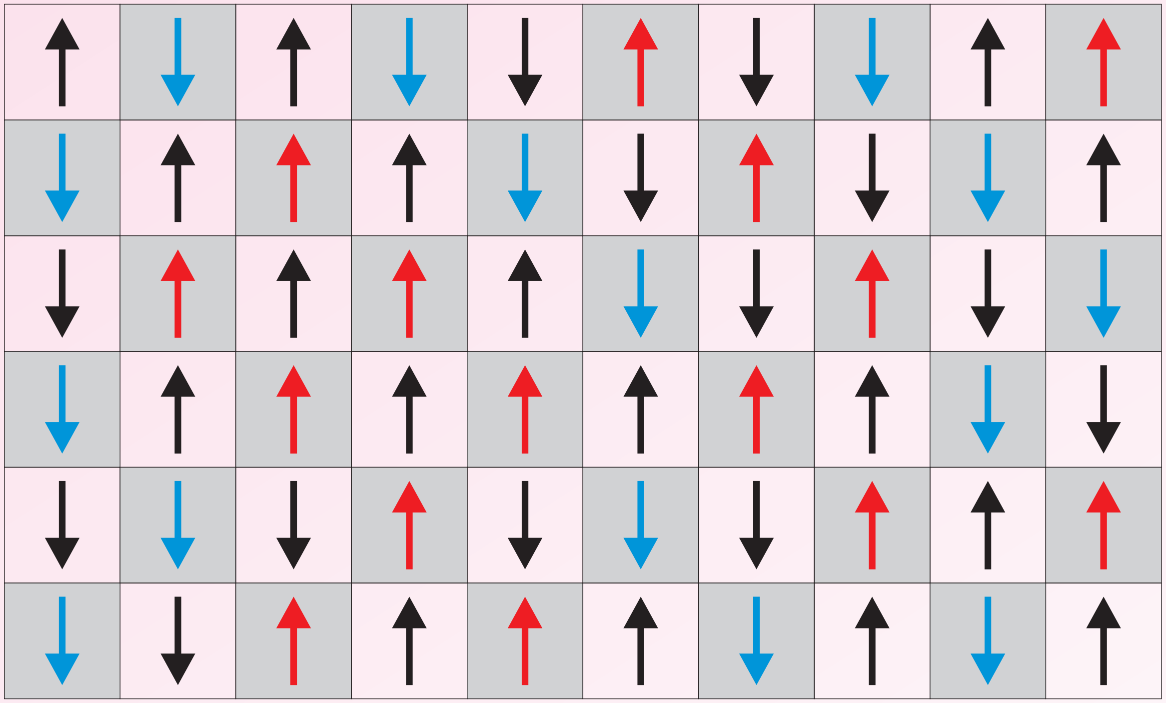

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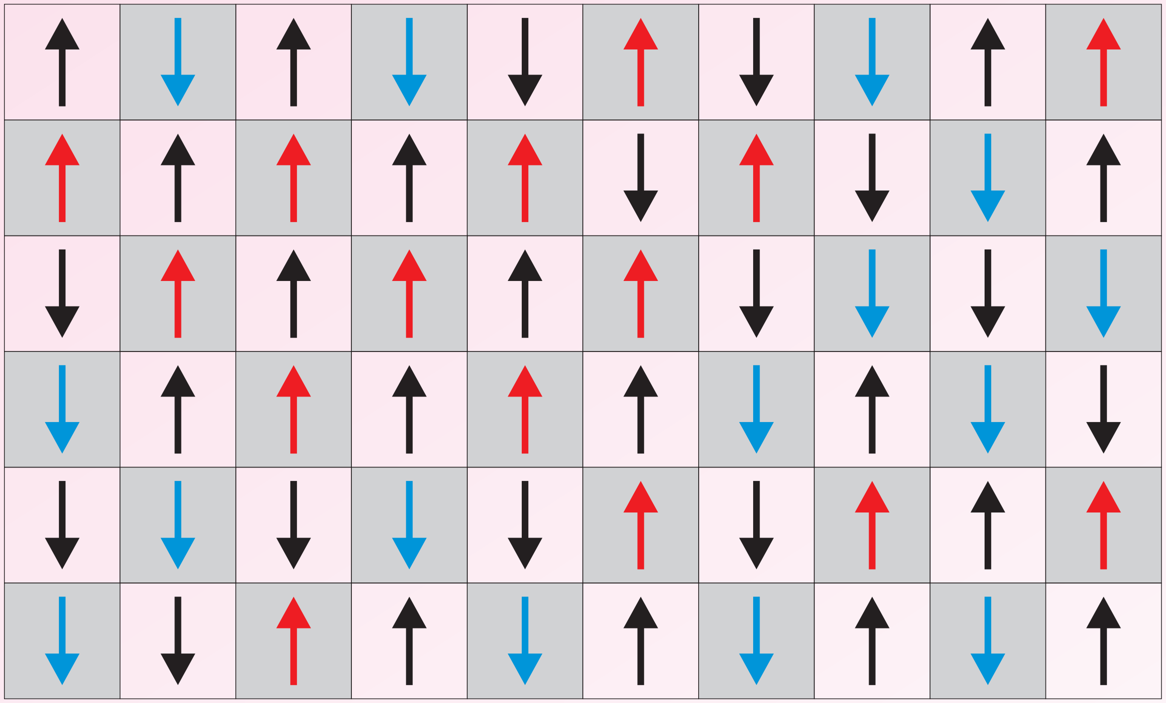
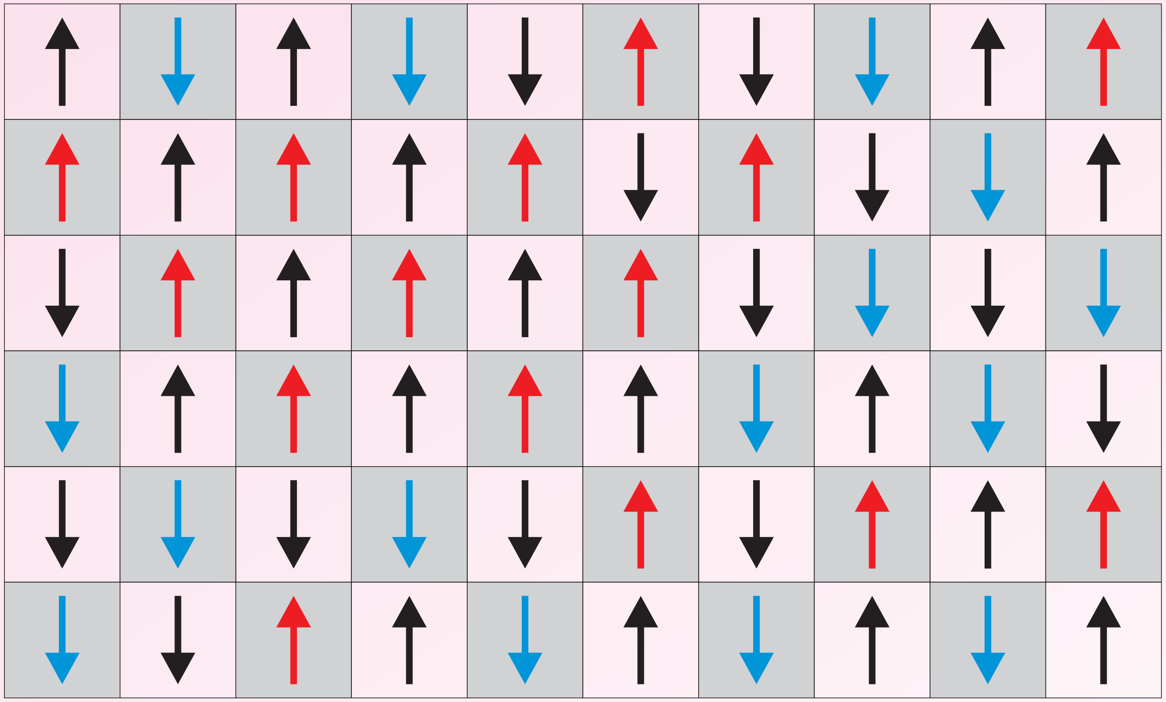
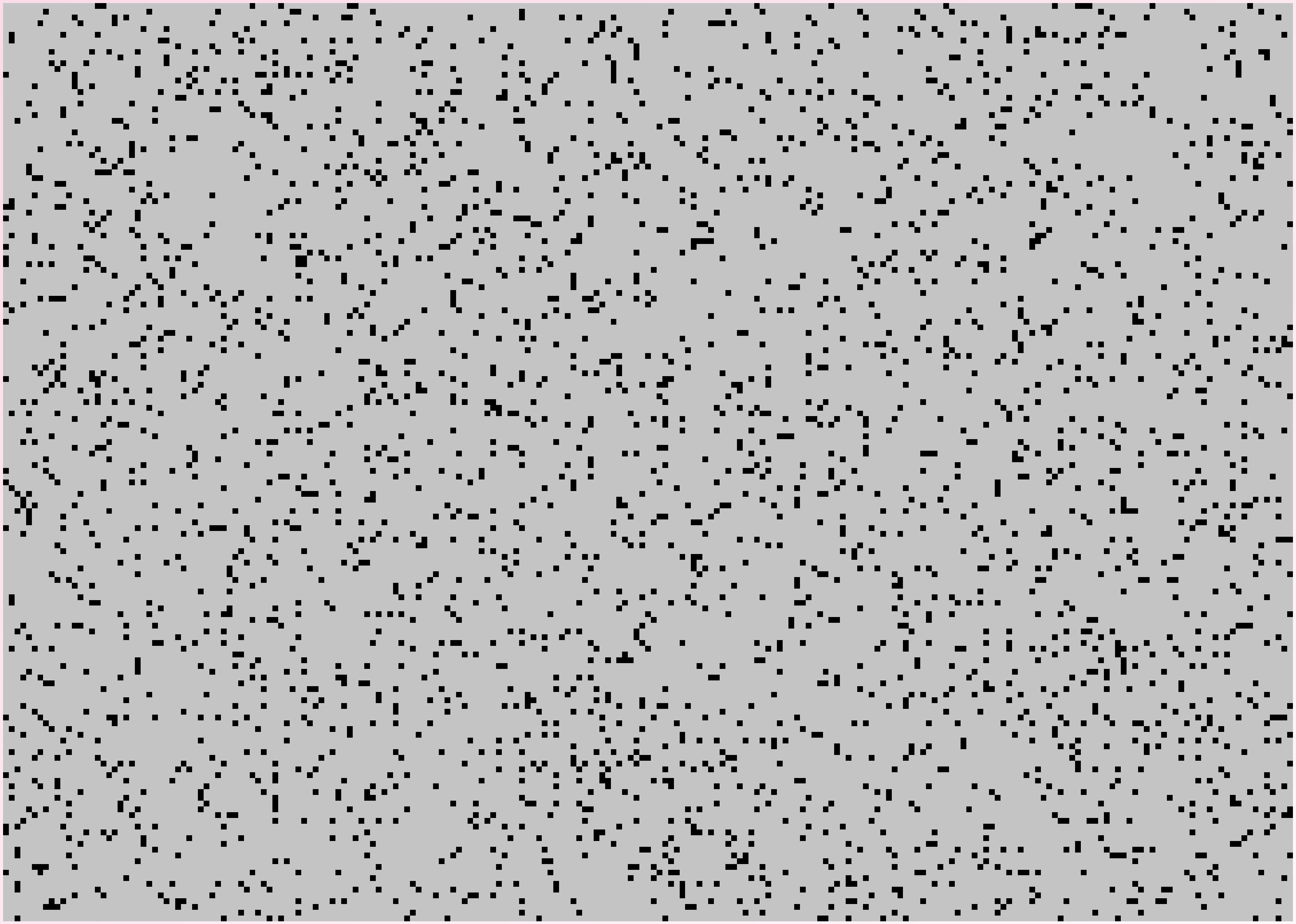


Q2R is **reversible**: The same rule (applied again on squares of the same color) reconstructs the previous generation.

Q2R rule also exhibits a local **conservation law**: The number of neighbors with opposite spins remains constant over time.

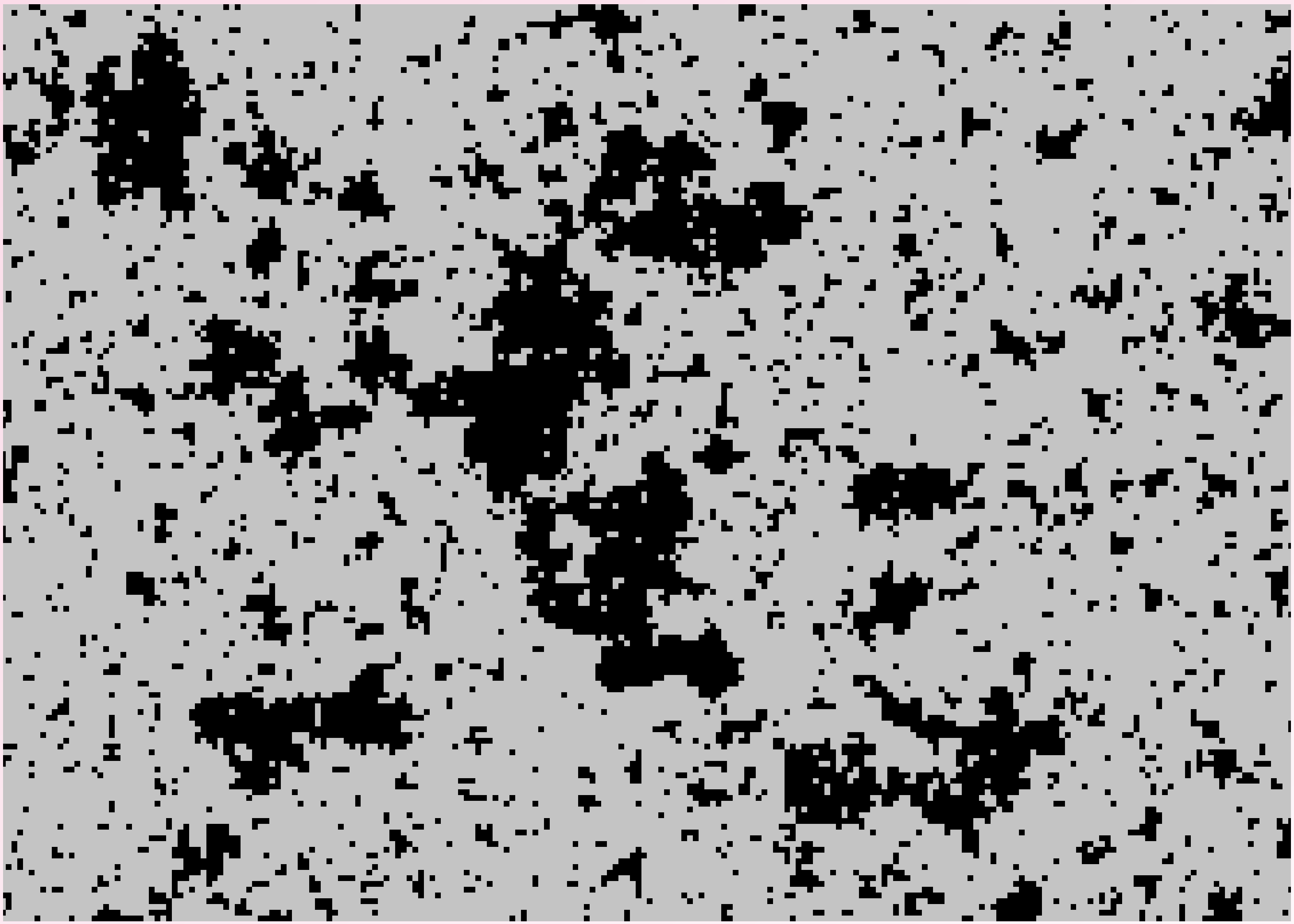


Evolution of Q2R from an uneven random distribution of spins:



Initial random configuration with 8% spins up.

Evolution of Q2R from an uneven random distribution of spins:

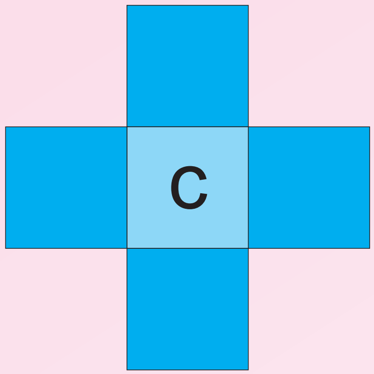


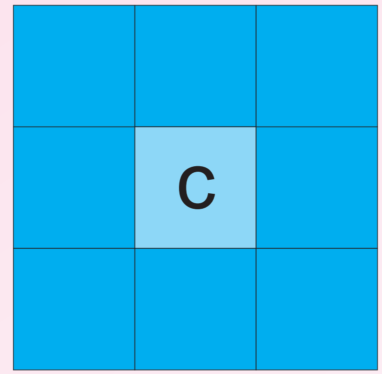
One million steps. The length of the B/W boundary is invariant.

General definition of d-dimensional CA

- Finite **state set** S.
- **Configurations** are elements of $S^{\mathbb{Z}^d}$, i.e., functions $\mathbb{Z}^d \longrightarrow S$ assigning states to cells,
- **Neighborhood** $N \subseteq \mathbb{Z}^d$ is a finite set of relative offsets to neighbors from each cell.
- The **neighbors** of a cell at location $\vec{x} \in \mathbb{Z}^d$ are the cells at locations $\vec{x} + \vec{n}$ for all $\vec{n} \in N$.

Typical two-dimensional neighborhoods:





Von Neumann

neighborhood

$\{(0,0),(\pm 1,0),(0,\pm 1)\}$

Moore

neighborhood

$\{-1,0,1\}\times\{-1,0,1\}$

The **local rule** is a function

$$f : S^N \longrightarrow S$$

where N is the neighborhood, providing the **new state** $f(p) \in S$ based on the pattern $p \in S^N$ that a cell sees in its neighbors.

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The **global dynamics** of the CA: Configuration c becomes in one time step the configuration e where, for all $\vec{x} \in \mathbb{Z}^d$,

$$e(\vec{x}) = f(c_{\vec{x}+N}).$$

The transformation

$$G : S^{\mathbb{Z}^d} \longrightarrow S^{\mathbb{Z}^d}$$

that maps $c \mapsto e$ is the **CA function**.

A CA is

- **injective** if G is one-to-one,
- **surjective** if G is onto,
- **bijective** if G is both one-to-one and onto.

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A CA G is a **reversible** (RCA) if there is another CA function F that is its inverse, i.e.

$$G \circ F = F \circ G = \text{identity function.}$$

RCA G and F are called the **inverse automata** of each other.

Curtis-Hedlund-Lyndon -theorem

It is convenient to endow $S^{\mathbb{Z}^d}$ with the usual **metric** to measures distances of configurations: For all $c \neq e$,

$$d(c, e) = 2^{-n}$$

where

$$n = \min\{||\vec{x}|| \mid c(\vec{x}) \neq e(\vec{x})\}$$

is the distance from the origin to the closest cell where c and e differ.

Two configurations are close to each other if one needs to look far to see a difference in them.

The metric induces a **compact** topology on $S^{\mathbb{Z}^d}$.

All cellular automata are **continuous** transformations

$$S^{\mathbb{Z}^d} \longrightarrow S^{\mathbb{Z}^d}$$

under our metric.

Indeed, locality of the update rule means that configurations that are close to each other have images close to each other.

The **translation** τ determined by vector $\vec{r} \in \mathbb{Z}^d$ is the transformation

$$S^{\mathbb{Z}^d} \longrightarrow S^{\mathbb{Z}^d}$$

that maps $c \mapsto e$ where

$$e(\vec{x}) = c(\vec{x} - \vec{r}) \text{ for all } \vec{x} \in \mathbb{Z}^d.$$

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Since all cells of a CA use the same local rule, the CA **commutes with all translations**:

$$G \circ \tau = \tau \circ G.$$

We have seen that all CA are continuous, translation commuting maps $S^{\mathbb{Z}^d} \longrightarrow S^{\mathbb{Z}^d}$.

The **Curtis-Hedlund- Lyndon theorem** from 1969 states that also the converse is true:

Theorem: A function $G : S^{\mathbb{Z}^d} \longrightarrow S^{\mathbb{Z}^d}$ is a CA function if and only if

(i) G is continuous, and

(ii) G commutes with translations.

From the Curtis-Hedlund-Lyndon -theorem we get

Corollary: A cellular automaton G is reversible if and only if it is bijective.

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Corollary: A cellular automaton G is reversible if and only if it is bijective.

Proof: $\Longrightarrow$ is trivial.

$\Longleftarrow$: Suppose that G is a bijective CA function. Then G has an inverse function G^{-1} that clearly commutes with the shifts. The inverse function G^{-1} is also continuous because the space $S^{\mathbb{Z}^d}$ is compact. It now follows from the Curtis-Hedlund-Lyndon theorem that G^{-1} is a cellular automaton. $\square$

From the Curtis-Hedlund-Lyndon -theorem we get

Corollary: A cellular automaton G is reversible if and only if it is bijective.

The **point of the corollary** is that in bijective CA each cell can determine its previous state by looking at the current states in some bounded neighborhood around them.

Some symbolic dynamics terminology:

- The set $S^{\mathbb{Z}^d}$ (together with translations) is the d-dimensional **full shift**.
- A subset of $S^{\mathbb{Z}^d}$ defined by forbidding some finite pattern is a **subshift**. These are precisely the topologically closed, translation invariant subsets of $S^{\mathbb{Z}^d}$.
- Cellular automata are the endomorphisms of the full shift.

Garden-Of-Eden and orphans

Configurations that do not have a pre-image are called **Garden-Of-Eden** -configurations. Only non-surjective CA have GOE configurations.

A finite pattern consists of a finite domain $D \subseteq \mathbb{Z}^d$ and an assignment

$$p : D \longrightarrow S$$

of states.

Finite pattern is called an **orphan** for CA G if every configuration containing the pattern is a GOE.

From the compactness of $S^{\mathbb{Z}^d}$ we directly get:

Proposition. Every GOE configuration contains an orphan pattern.

Non-surjectivity is hence equivalent to the existence of orphans.

Balance in surjective CA

All surjective CA have **balanced** local rules: for every $a \in S$

$$\left|f^{-1}(a)\right| = |S|^{n-1}.$$

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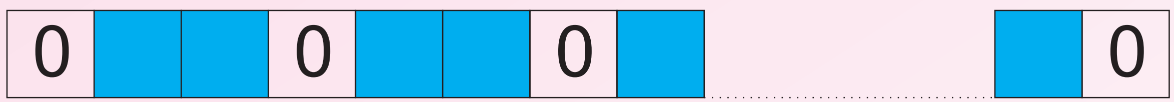
Indeed, consider a non-balanced local rule such as rule 110 where five contexts give new state 1 while only three contexts give state 0:

$$\begin{array}{ccc}
111 & \longrightarrow & 0 \\
110 & \longrightarrow & 1 \\
101 & \longrightarrow & 1 \\
100 & \longrightarrow & 0 \\
011 & \longrightarrow & 1 \\
010 & \longrightarrow & 1 \\
001 & \longrightarrow & 1 \\
000 & \longrightarrow & 0
\end{array}$$

Consider finite patterns where state 0 appears in every third position. There are $2^{2(k-1)} = 4^{k-1}$ such patterns where k is the number of 0's.



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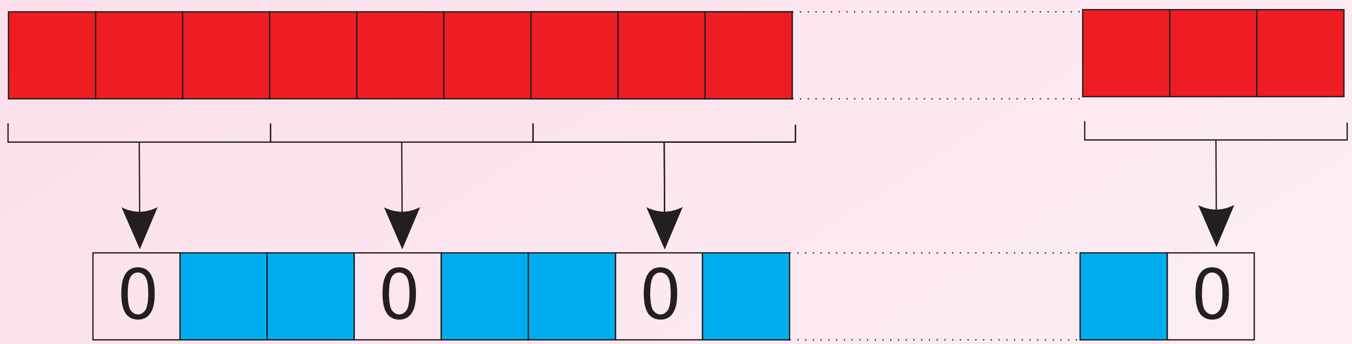


A pre-image of such a pattern must consist of k segments of length three, each of which is mapped to 0 by the local rule. There are 3^k choices.

As for large values of k we have $3^k < 4^{k-1}$, there are fewer choices for the red cells than for the blue ones. Hence some pattern has no pre-image and it must be an orphan. $\square$

One can also verify directly that pattern

$$01010$$

is an orphan of rule 110. It is the shortest orphan.

Balance of the local rule is not sufficient for surjectivity. For example, the **majority** CA (Wolfram number 232) is a counter example. The local rule

$$f(a, b, c) = 1 \text{ if and only if } a + b + c \geq 2$$

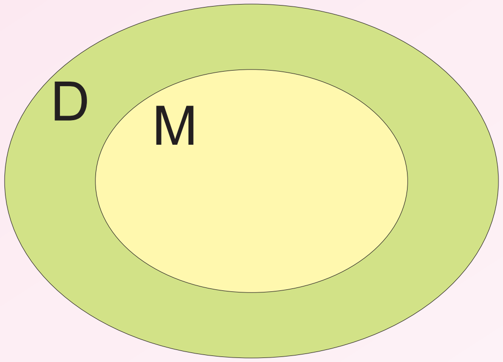
is clearly balanced, but 01001 is an orphan.

The balance property of surjective CA generalizes to finite patterns of arbitrary shape:

Theorem: Let G be surjective. Let $M, D \subseteq \mathbb{Z}^d$ be finite domains such that D contains the neighborhood of M. Then every finite pattern with domain M has the same number

$$n^{|D|-|M|}$$

of pre-images in domain D, where n is the number of states. $\square$



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The balance property means that the uniform probability measure is **invariant** for surjective CA. (Uniform randomness is preserved by surjective CA.)

Garden-Of-Eden -theorem

Let us call configurations c_1 and c_2 **asymptotic** if the set

$$\mathit{diff}\,(c_1, c_2) = \{\vec{n} \in \mathbb{Z}^d \mid c_1(\vec{n}) \neq c_2(\vec{n})\ \}$$

of positions where c_1 and c_2 differ is finite.

A CA is called **pre-injective** if any asymptotic $c_1 \neq c_2$ satisfy $G(c_1) \neq G(c_2)$.

The **Garden-Of-Eden -theorem** by Moore (1962) and Myhill (1963) connects surjectivity with pre-injectivity.

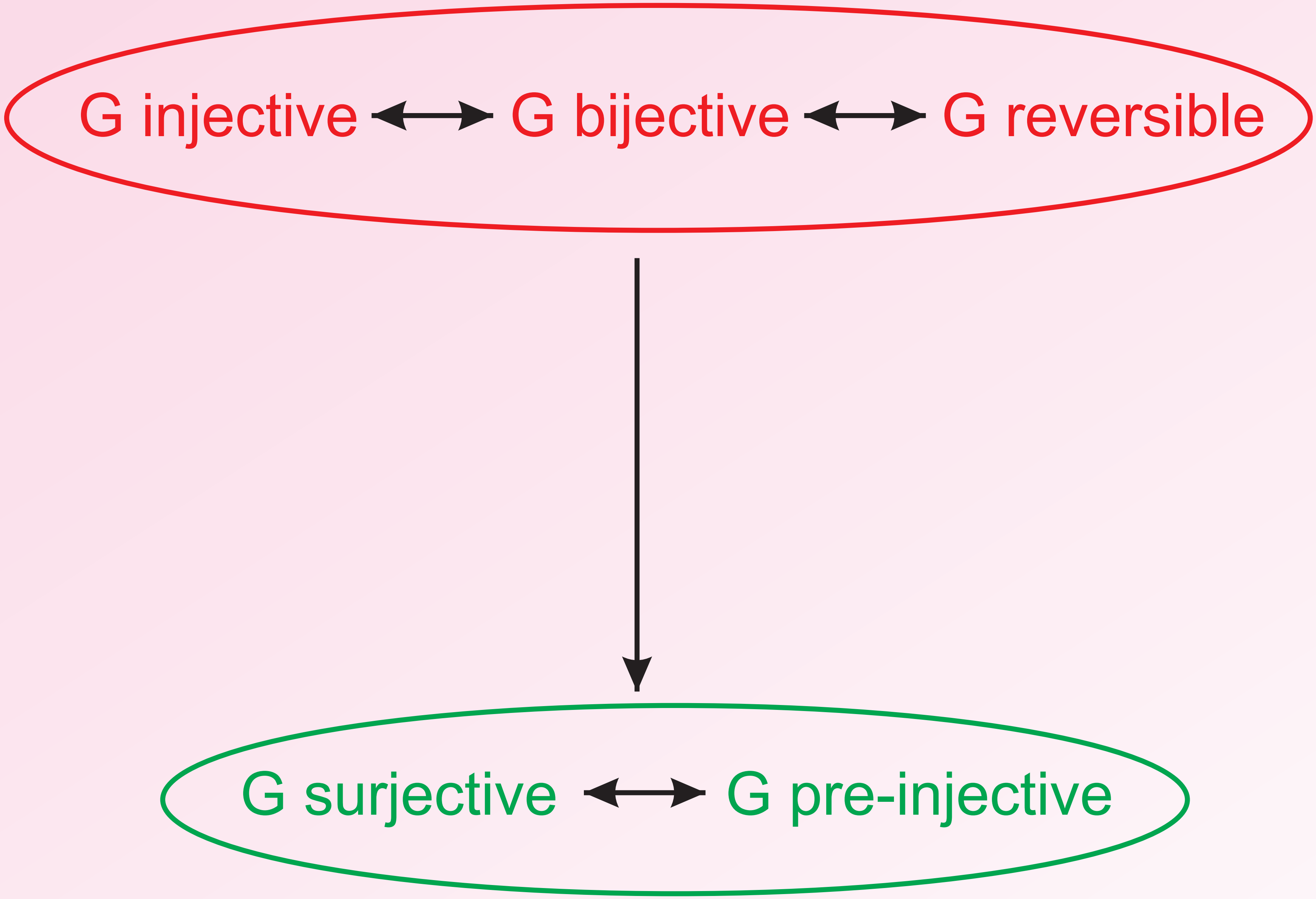
Theorem: CA G is surjective if and only if it is pre-injective.

The **Garden-Of-Eden -theorem** by Moore (1962) and Myhill (1963) connects surjectivity with pre-injectivity.

Theorem: CA G is surjective if and only if it is pre-injective.

Corollary: Every injective CA is also surjective. Injectivity, bijectivity and reversibility are equivalent concepts.

Proof: If G is injective then it is pre-injective. The claim follows from the Garden-Of-Eden -theorem. $\square$


G injective
G bijective
G reversible
G surjective
G pre-injective

Examples:

The majority rule is not surjective: asymptotic configurations

$$\ldots 0000000 \ldots \qquad \text{and} \qquad \ldots 0001000 \ldots$$

have the same image, so G is not pre-injective. Pattern

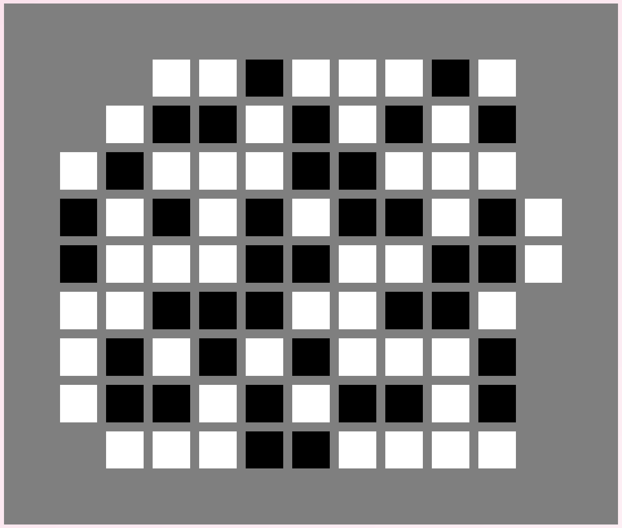
$$01001$$

is an orphan.

Examples:

In Game-Of-Life a lonely living cell dies immediately, so G is not pre-injective. GOL is hence not surjective.

Interestingly, no small orphans are known for Game-Of-Life. Currently, the smallest known orphan consists of 88 cells (50 life=white, 38 dead=black):



Steven Eker (2017)

Examples:

The **Traffic CA** is the elementary CA number 226.

$$\begin{array}{ccc} 111 & \longrightarrow & 1 \\ 110 & \longrightarrow & 1 \\ 101 & \longrightarrow & 1 \\ 100 & \longrightarrow & 0 \\ 011 & \longrightarrow & 0 \\ 010 & \longrightarrow & 0 \\ 001 & \longrightarrow & 1 \\ 000 & \longrightarrow & 0 \end{array}$$

The local rule replaces pattern 01 by pattern 10.

$$
\begin{array}{ccc}
111 & \longrightarrow & 1 \\
110 & \longrightarrow & 1 \\
101 & \longrightarrow & 1 \\
100 & \longrightarrow & 0 \\
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\end{array}
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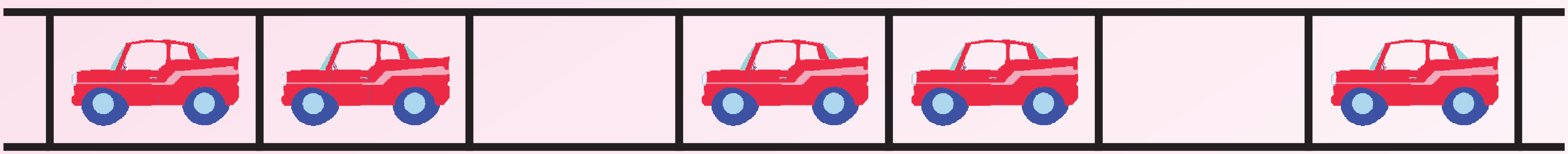

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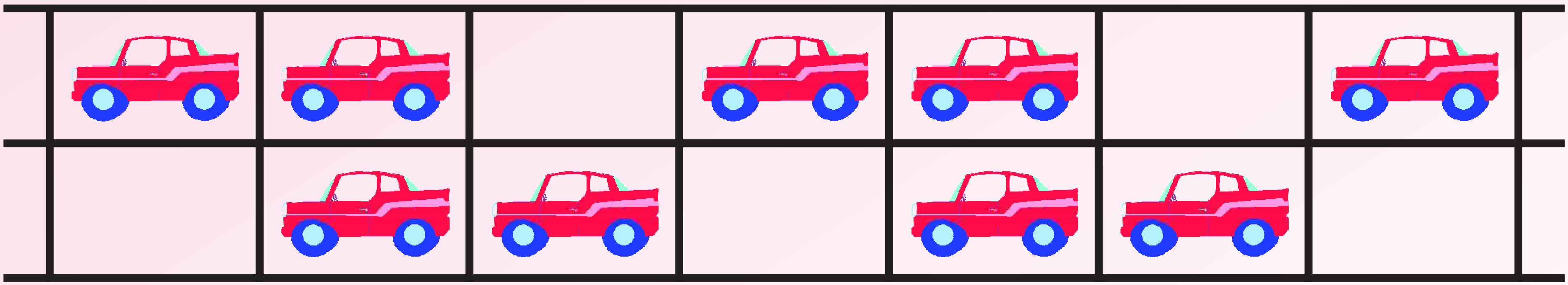
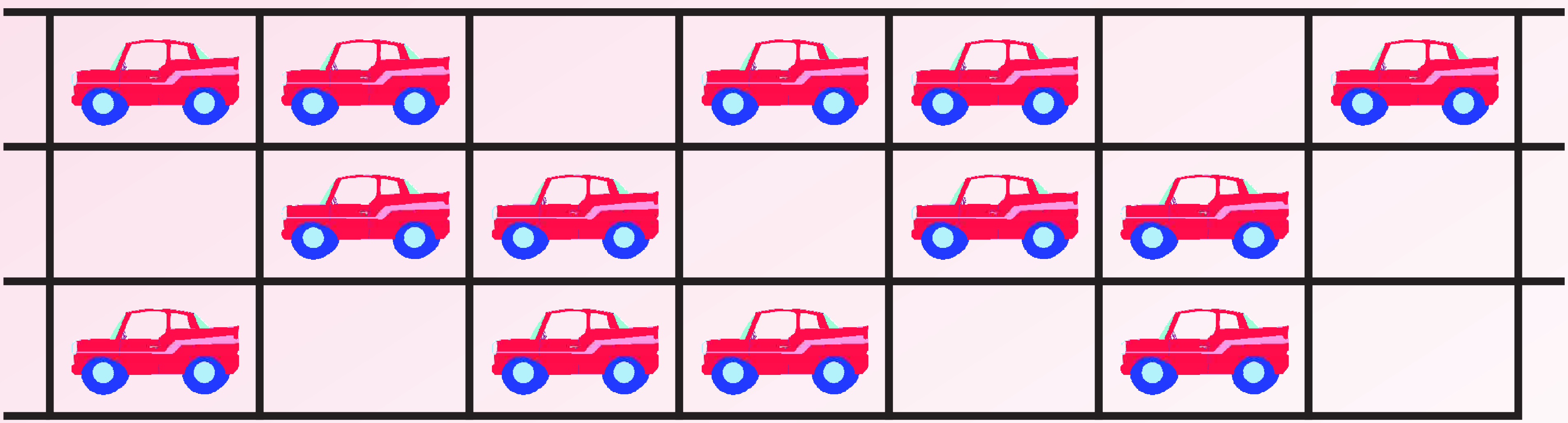
\end{array}
$$



$$
\begin{array}{lcl}
111 & \longrightarrow & 1 \\
110 & \longrightarrow & 1 \\
101 & \longrightarrow & 1 \\
100 & \longrightarrow & 0 \\
011 & \longrightarrow & 0 \\
010 & \longrightarrow & 0 \\
001 & \longrightarrow & 1 \\
000 & \longrightarrow & 0
\end{array}
$$



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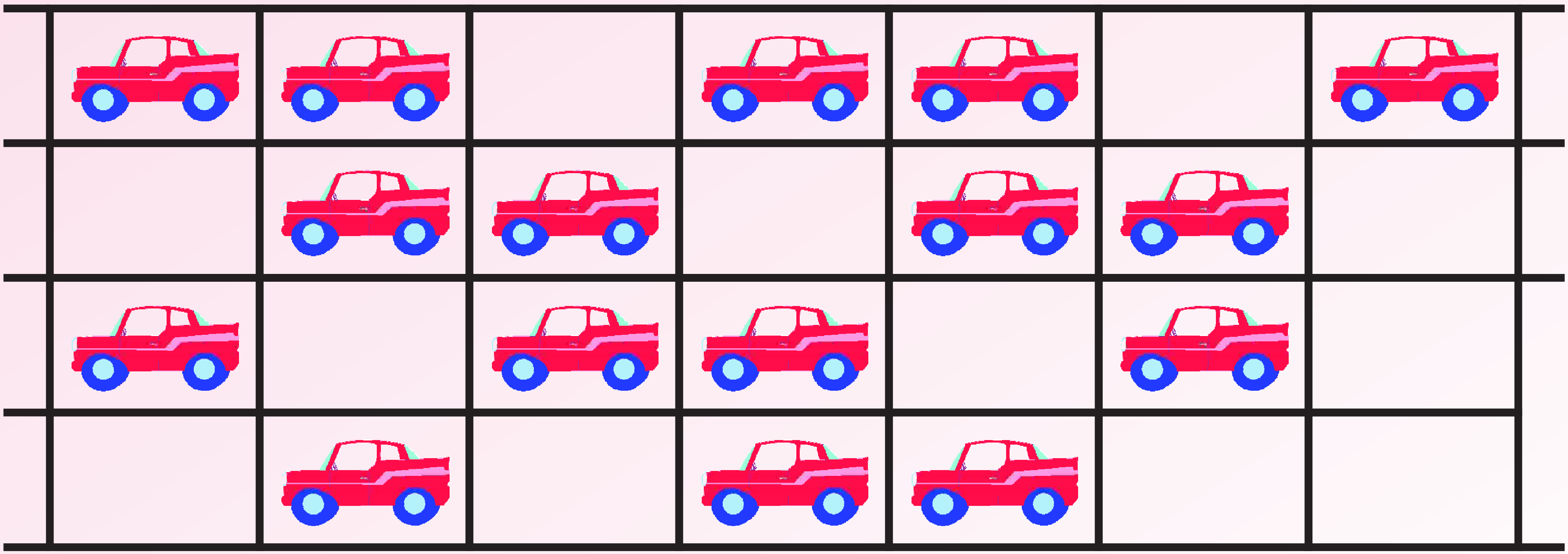
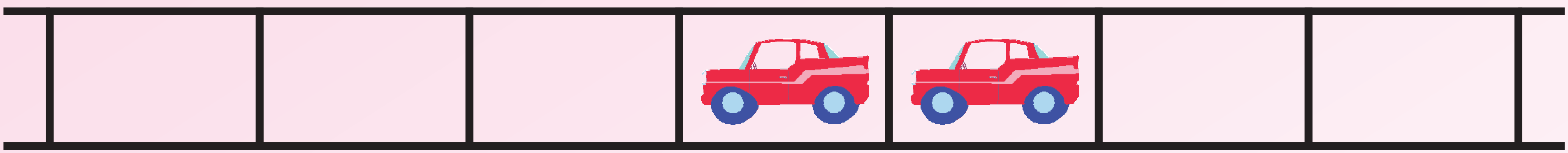
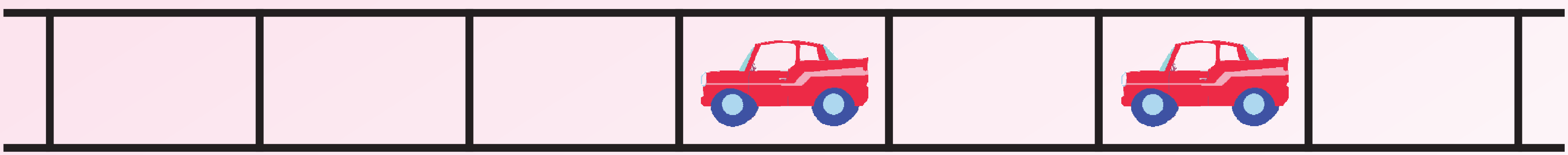


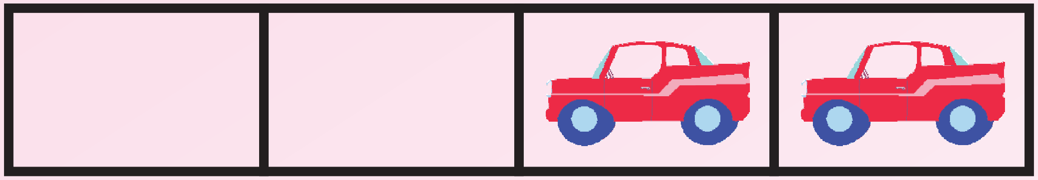
The local rule is balanced. However, there are two asymptotic configurations with the same successor:

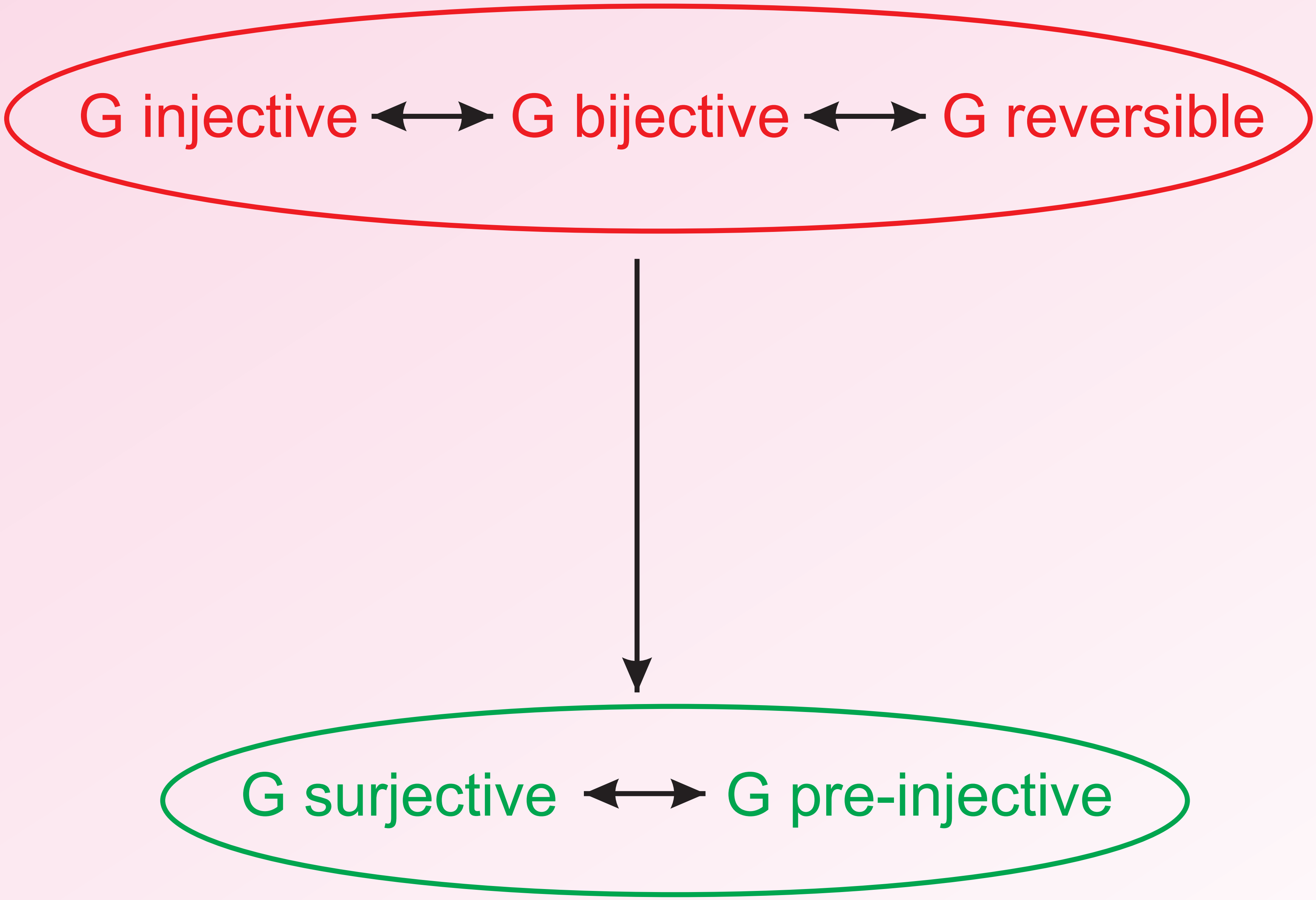


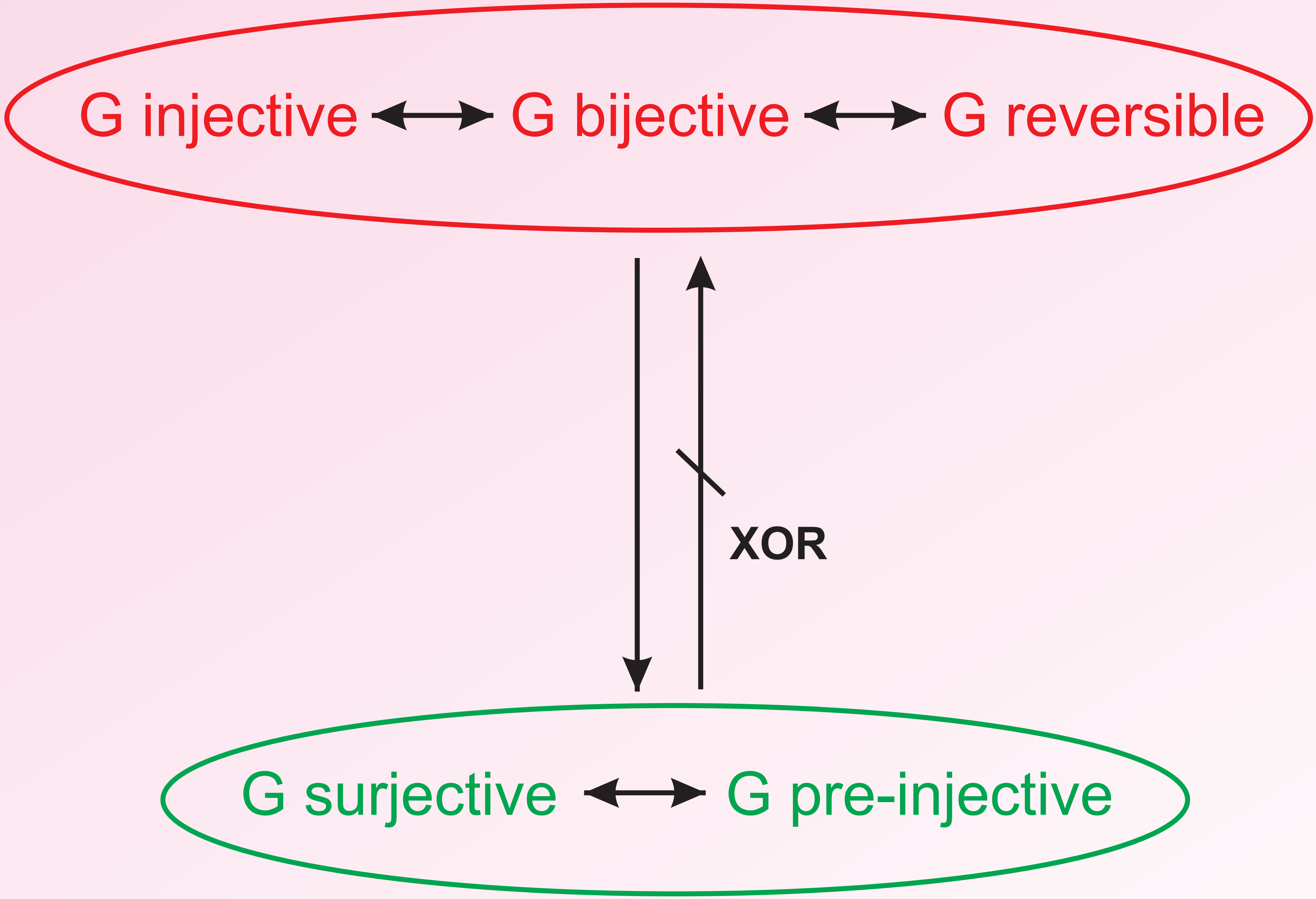


and hence traffic CA is not surjective.

There is an orphan of size four:




G injective
G bijective
G reversible
G surjective
G pre-injective


G injective
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G reversible
XOR
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The xor-CA is the binary state CA with neighborhood $(0,1)$ and local rule

$$f(a,b) = a + b \pmod 2.$$

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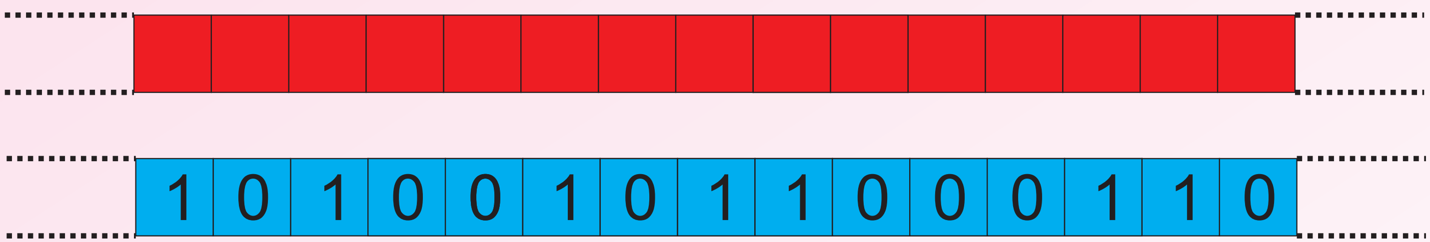
In the xor-CA every configuration has exactly two pre-images, so G is surjective but not injective:



One can freely choose one value in the pre-image, after which all remaining states are uniquely determined by the **left-permutativity** and the **right-permutativity** of xor.

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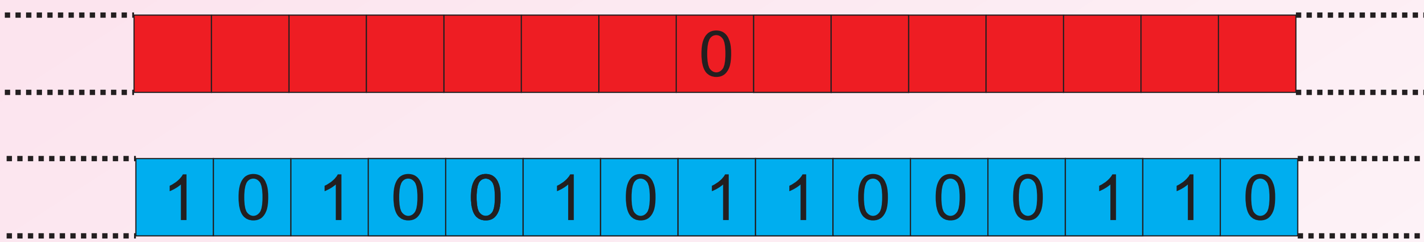

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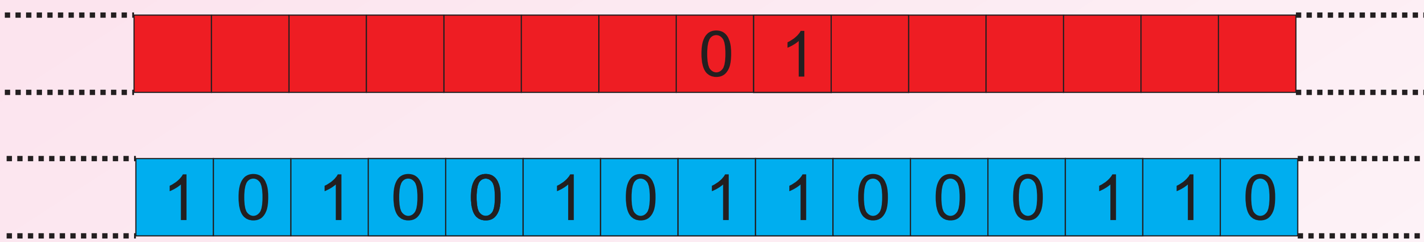

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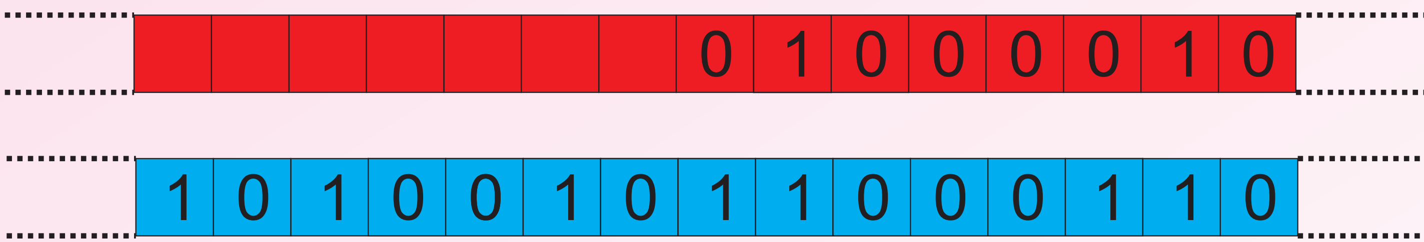

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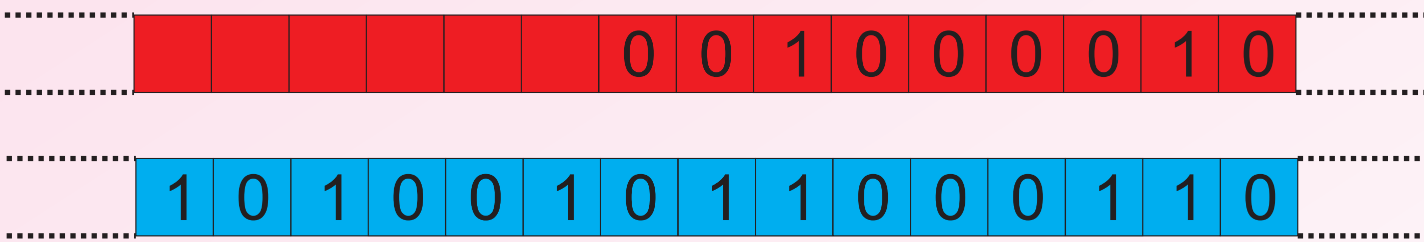

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1	0	0	1	1	1	0	0	1	0	0	0	0	1	0
1	0	1	0	0	1	0	1	1	0	0	0	1	1	0

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Periodic configurations

It is obviously not possible to simulate CA functions on arbitrary infinite configurations, but one has to limit the attention to some subset of $S^{\mathbb{Z}^d}$.

We often consider the action on **periodic configurations**.

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We often consider the action on **periodic configurations**.

Cellular automata preserve periods, so periodic configurations are mapped to periodic configurations.

The use of periodic configurations is usually termed **periodic boundary conditions**.

Periodic configurations are **dense** in the metric space $S^{\mathbb{Z}^d}$.

Surjectivity and injectivity of G_P

Let G_P denote the restriction of cellular automaton G on (fully) periodic configurations.

Implications

$$\begin{array}{rcl} \mathbf{G} \text{ injective} & \Longrightarrow & \mathbf{G_P} \text{ injective} \\ \mathbf{G_P} \text{ surjective} & \Longrightarrow & \mathbf{G} \text{ surjective} \end{array}$$

are easy. (Second one uses denseness of periodic configurations in $S^{\mathbb{Z}^d}$.)

We also have

$$\mathbf{G_P} \text{ injective} \implies \mathbf{G_P} \text{ surjective}$$

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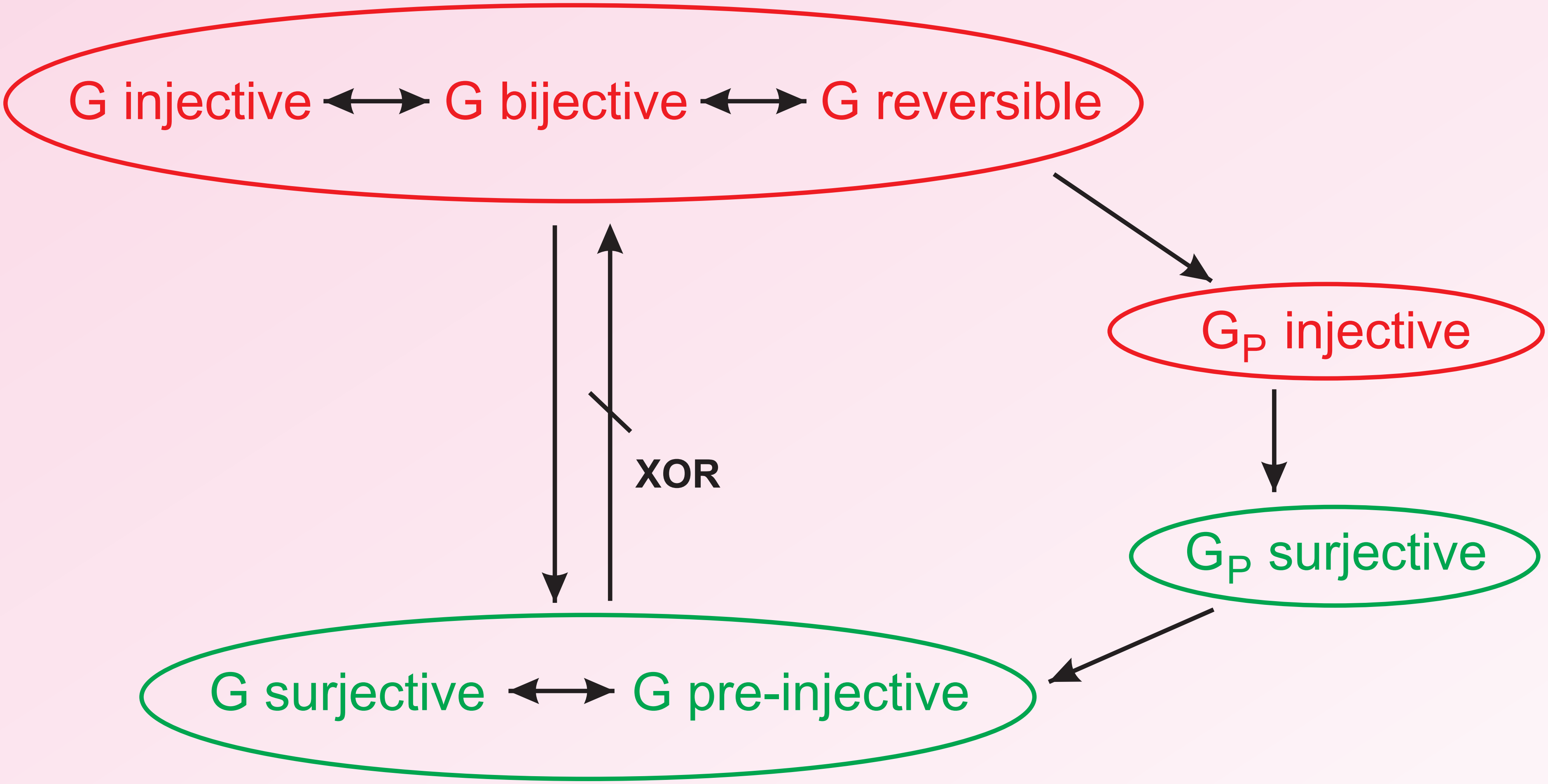
Indeed, fix any d linearly independent periods, and let $A \subseteq S^{\mathbb{Z}^d}$ be the set of configurations with these periods. Then

- A is finite,
- G is injective on A,
- $G(A) \subseteq A$.

We conclude that $G(A) = A$, and every periodic configuration has a periodic pre-image.


G injective
G bijective
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XOR
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Here we get the first **dimension sensitive** property. The following equivalences are only known to hold among one-dimensional CA:

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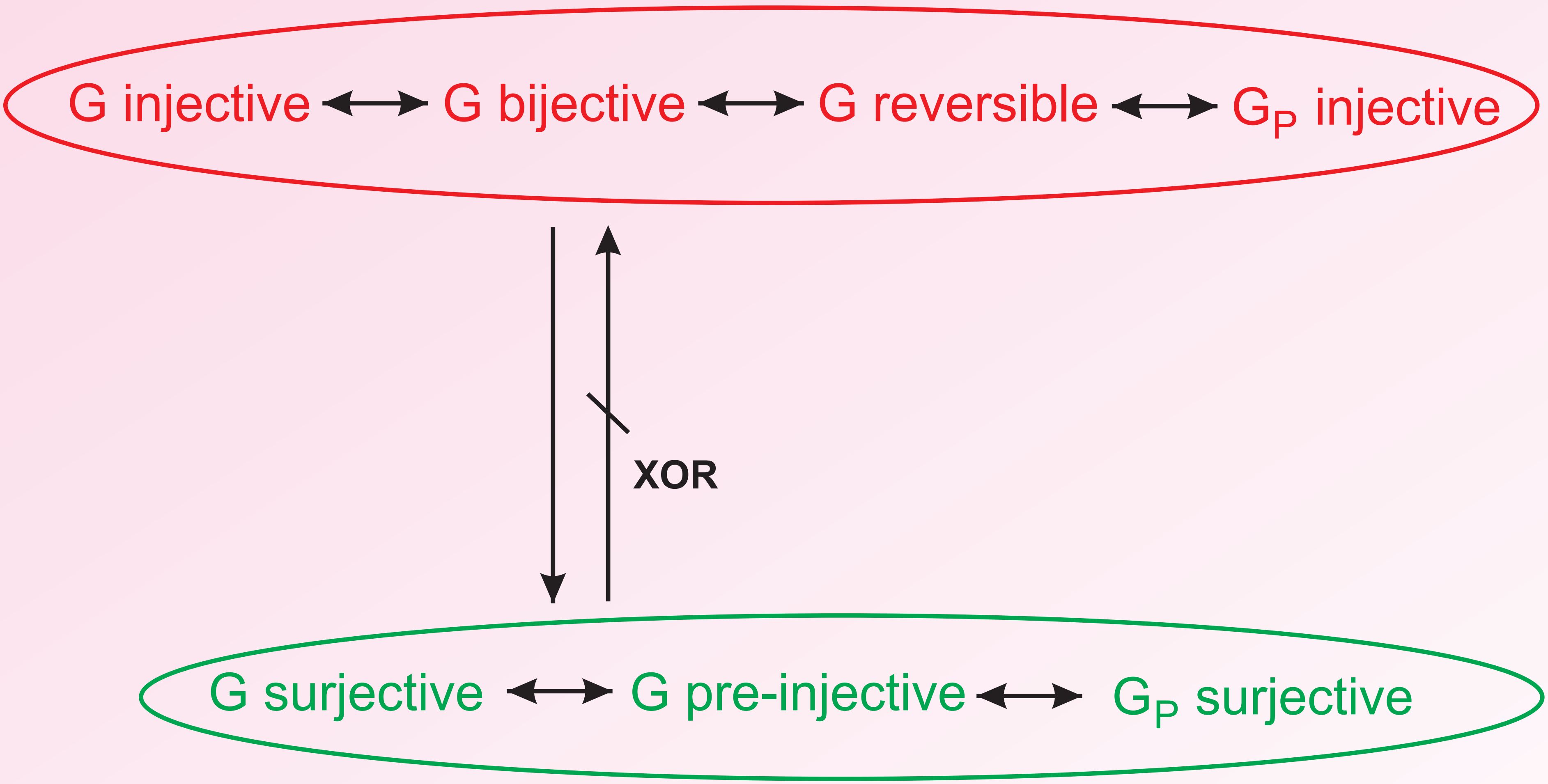
$$\begin{aligned} \mathbf{G} \text{ injective} &\iff \mathbf{G_P} \text{ injective} \\ \mathbf{G} \text{ surjective} &\iff \mathbf{G_P} \text{ surjective} \end{aligned}$$

- The first equivalence is not true among two-dimensional CA: Later we'll see a counter example **Snake-XOR**.
- It is not known whether the second equivalence is true in 2D.

Only in 1D



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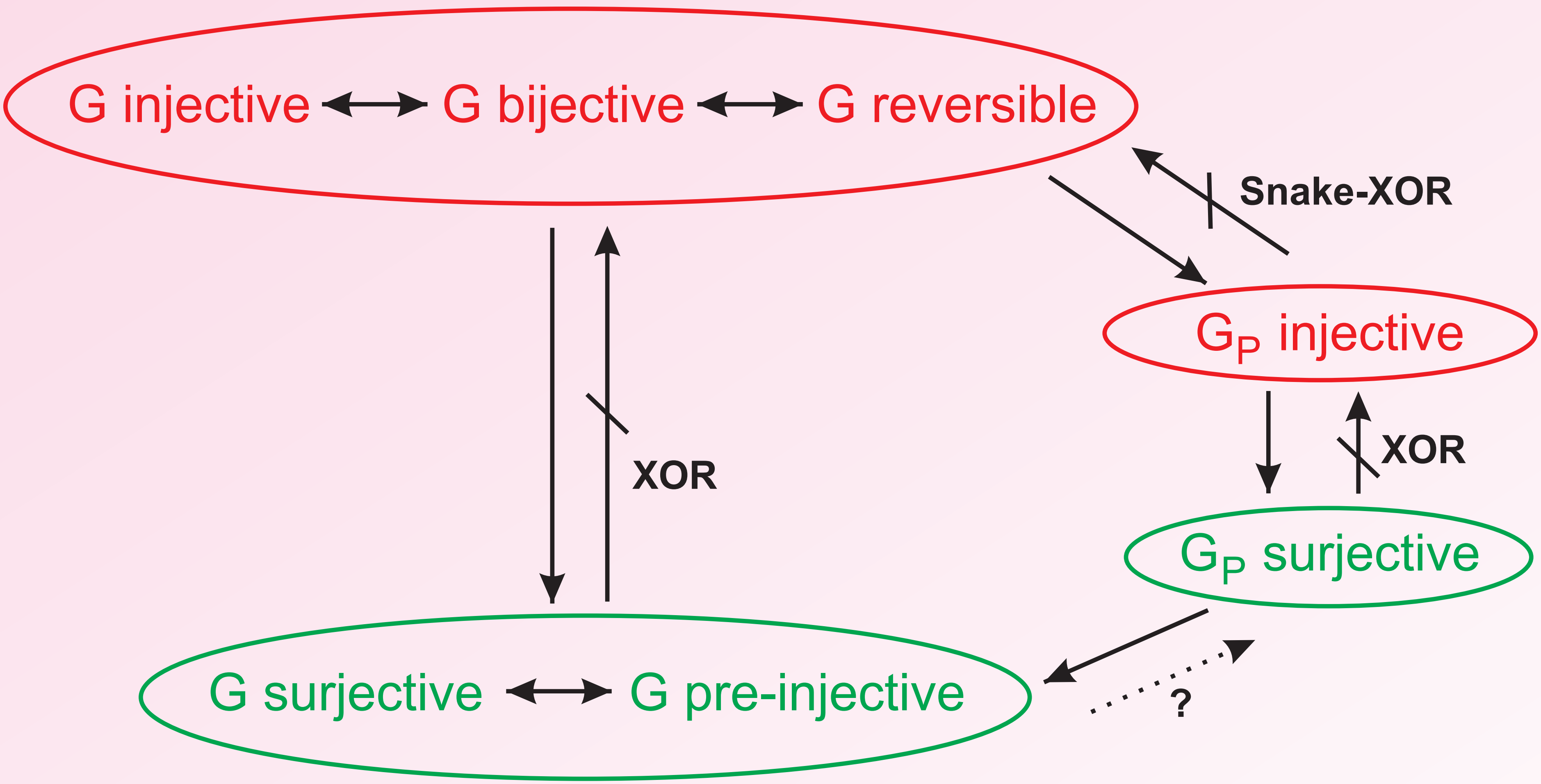
In 2D



G injective
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Snake-XOR
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?
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We have two proofs that injective CA are surjective:

$\mathbf{G}$ injective $\Longrightarrow$ $\mathbf{G}$ pre-injective $\Longrightarrow$ $\mathbf{G}$ surjective

$\mathbf{G}$ injective $\Longrightarrow$ $\mathbf{G_P}$ injective $\Longrightarrow$ $\mathbf{G_P}$ surjective $\Longrightarrow$ $\mathbf{G}$ surjective

We have two proofs that injective CA are surjective:

$$\mathbf{G} \text{ injective} \implies \mathbf{G} \text{ pre-injective} \implies \mathbf{G} \text{ surjective}$$

$$\mathbf{G} \text{ injective} \implies \mathbf{G_P} \text{ injective} \implies \mathbf{G_P} \text{ surjective} \implies \mathbf{G} \text{ surjective}$$

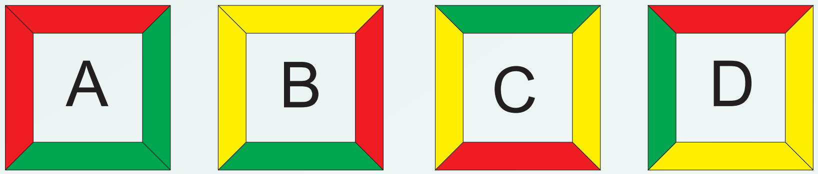
It is good to have both implication chains available, if one wants to generalize results to cellular automata whose underlying grid is not $\mathbb{Z}^d$ but some other group.

- The first chain generalizes to all **amenable** groups.
- The second chain generalizes to **residually finite** groups.

A group is called **surjunctive** if every injective CA on the group is also surjective. It is not known if all groups are surjunctive.

From tiles to cellular automata

Happened so far:



The **tiling problem** (or the **Domino problem**): Does a given Wang tile set admit a tiling of the plane?

Theorem (R.Berger 1966): The tiling problem of Wang tiles is undecidable.

Valid tilings form a subshift (of finite type):

topologically closed, translation invariant set,

defined by a finite number of forbidden local patterns.

Cellular automata:

continuous, translation commuting maps.

Wang tilings are "static" versions of "dynamic" 2D cellular automata

$\Longrightarrow$ undecidability results for 2D CA.

Example: It is undecidable whether a given two-dimensional CA G has any fixed point configurations.

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Proof: Reduction from the tiling problem.
For any given Wang tile set T (with at least two tiles), construct a two-dimensional CA with

- state set T,
- the von Neumann -neighborhood,
- the local update rule that keeps a tile unchanged if and only if its colors match with the neighboring tiles.

Trivially, $G(c) = c$ if and only if c is a valid tiling. $\square$

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Trivially, $G(c) = c$ if and only if c is a valid tiling. $\square$

Note: For one-dimensional CA it is easily decidable whether fixed points exist.

More interesting reduction: A CA is called **nilpotent** if all configurations eventually evolve into a fixed quiescent (=all states in same ground state q) configuration.

Observation: In a nilpotent CA all configurations must become quiescent within some uniformly bounded time n.

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Observation: In a nilpotent CA all configurations must become quiescent within some uniformly bounded time n.

Proof: A configuration that contains all finite patterns exists. It becomes quiescent at some time $n \Longrightarrow$ all configurations are quiescent at time n.

Theorem (Culik, Pachl, Yu, 1989): It is undecidable whether a given two-dimensional CA is nilpotent.

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Proof: For any given set T of Wang tiles we construct a two-dimensional CA that is nilpotent if and only if T does not admit a tiling.

For tile set T we make the following CA:

- State set is $S = T \cup \{q\}$ where q is a new symbol $q \notin T$,

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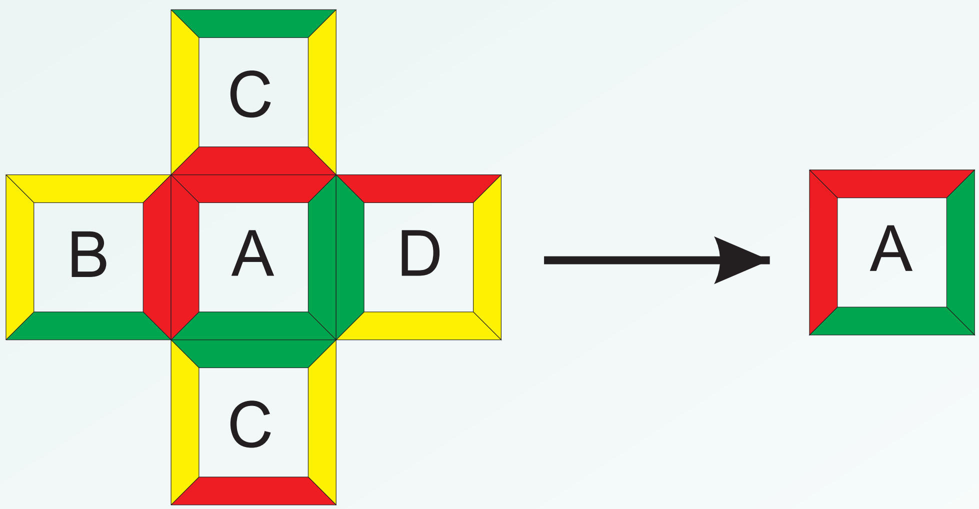

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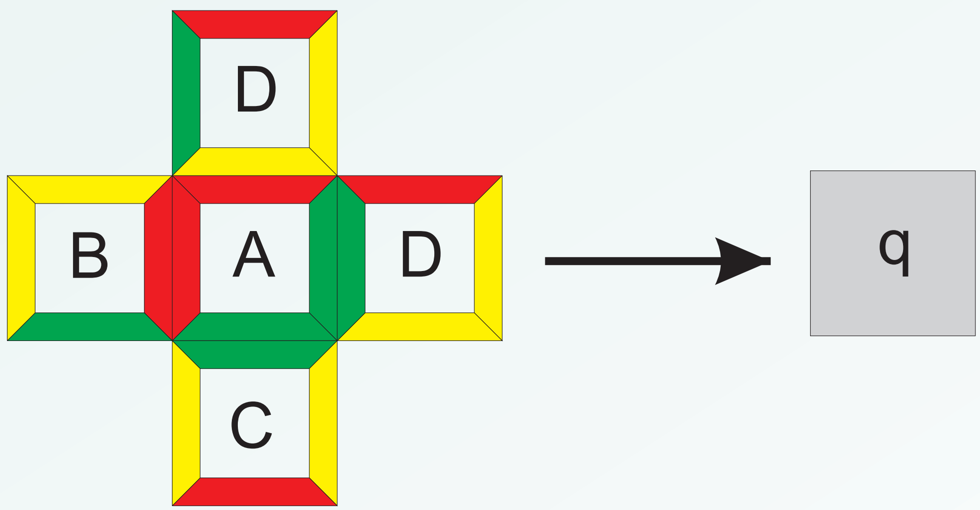

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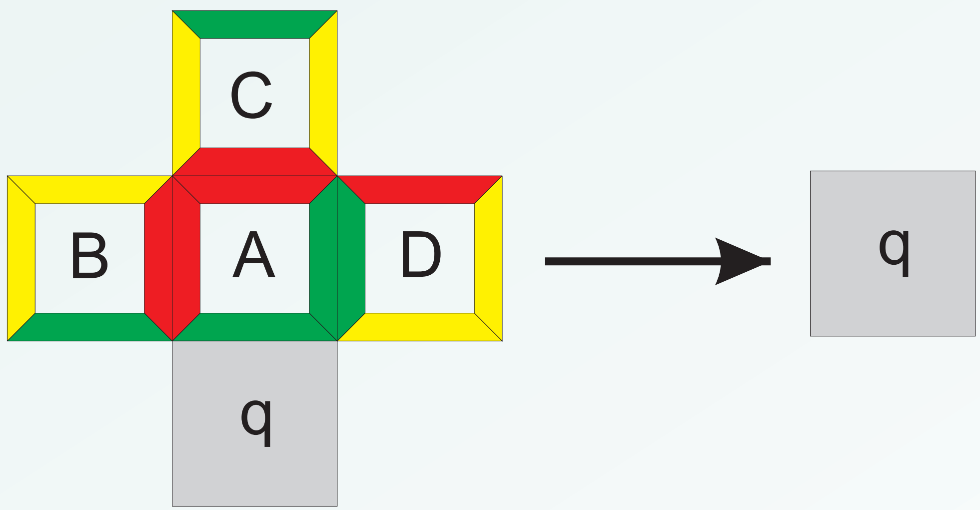

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$\Longleftarrow$ If T does not admit a valid tiling then every $n \times n$ square contains a tiling error, for some n. State q propagates, so in at most $2n$ steps all cells are in state q. The CA is nilpotent. $\square$

If we do the previous construction for an aperiodic tile set T we obtain a two-dimensional CA in which

- every periodic configuration becomes eventually quiescent, but
- there are some non-periodic fixed points.

What about 1D CA ?

Trick: view **space-time diagrams** as tilings. This imposes a determinism constraint on the considered tiles.

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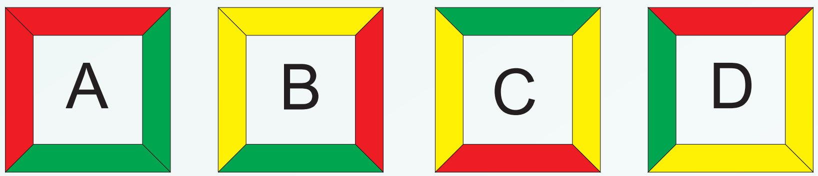
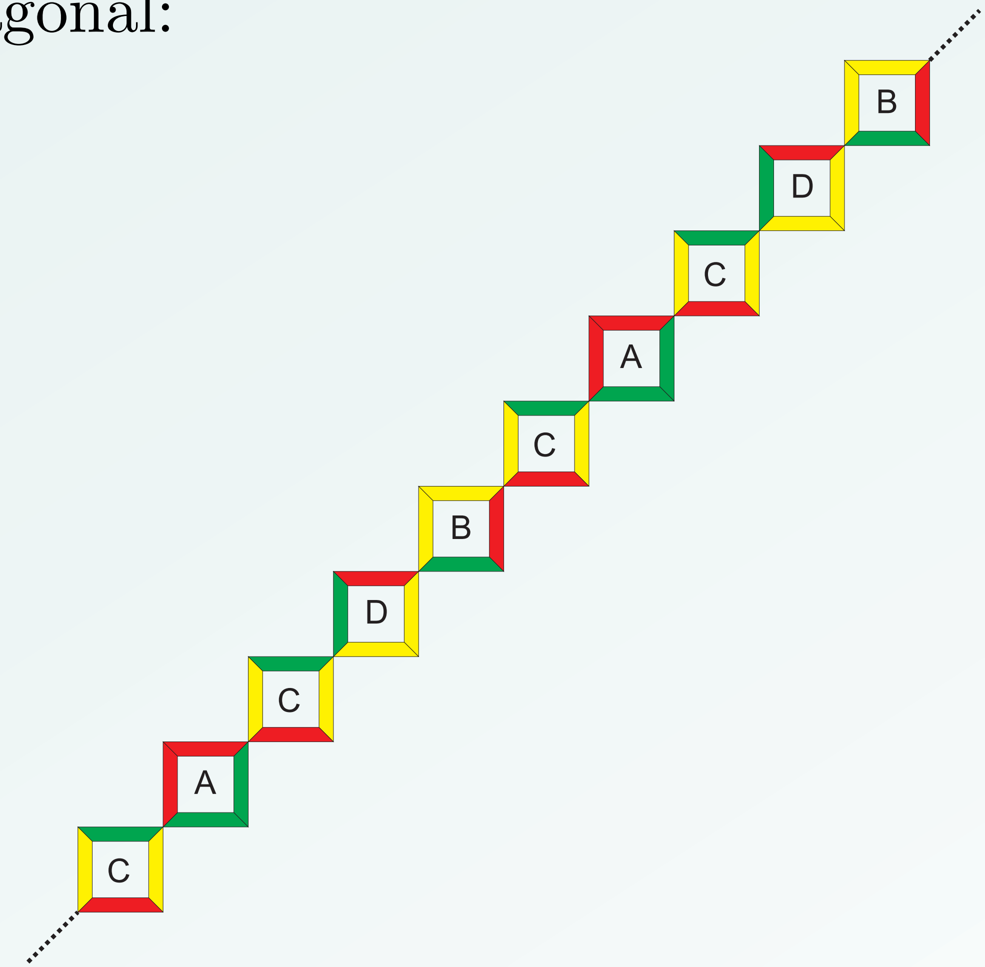
Tile set T is **NW-deterministic** if no two tiles have identical colors on their top edges and on their left edges. In a valid tiling the left and the top neighbor of a tile uniquely determine the tile.

For example, our sample tile set



is NW-deterministic.

In any valid tiling by NW-deterministic tiles, NE-to-SW diagonals uniquely determine the next diagonal below them. The tiles of the next diagonal are determined locally from the previous diagonal:



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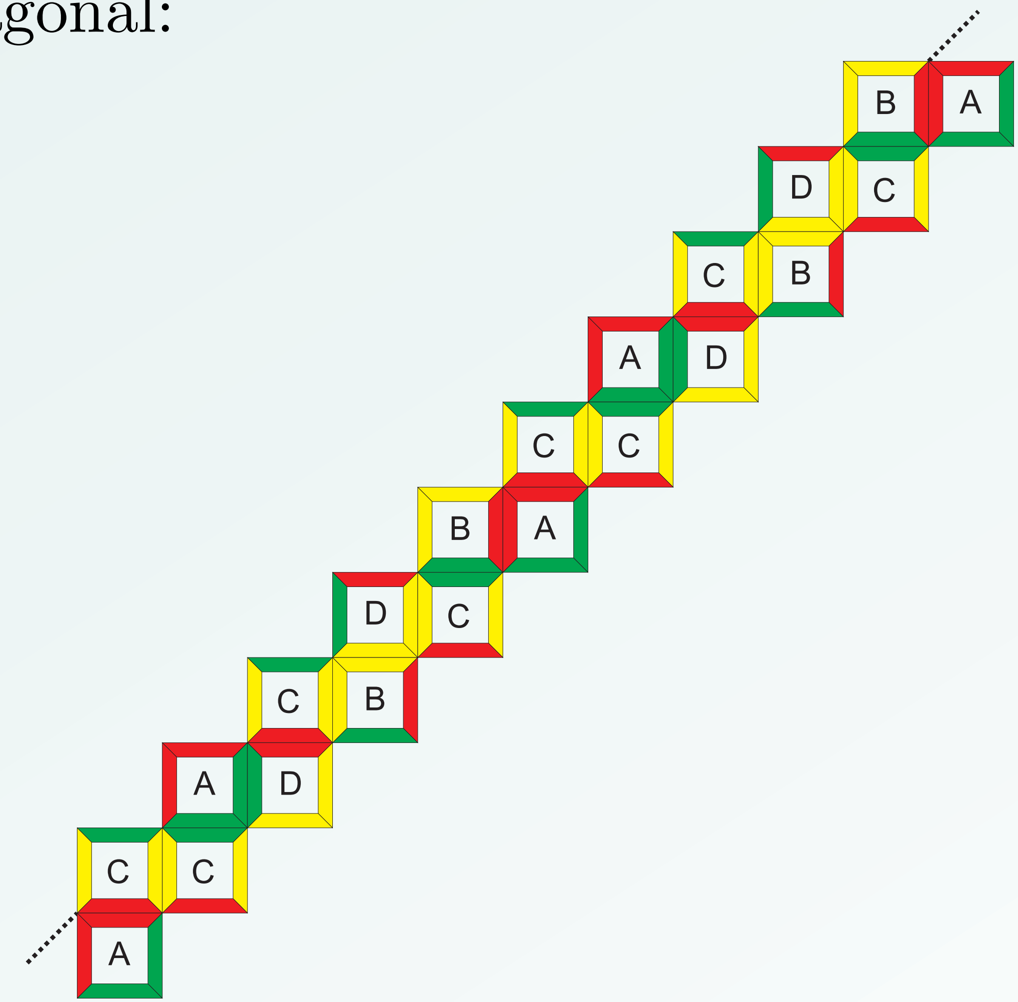

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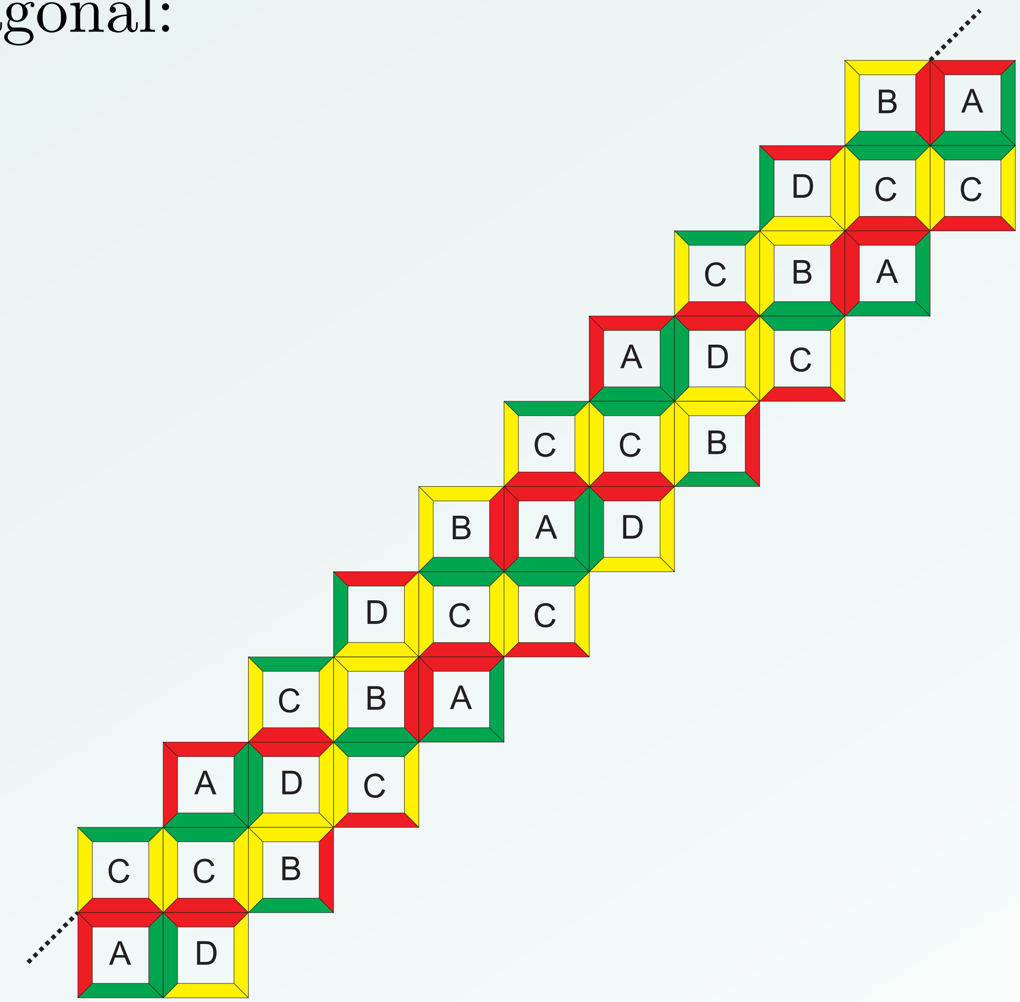

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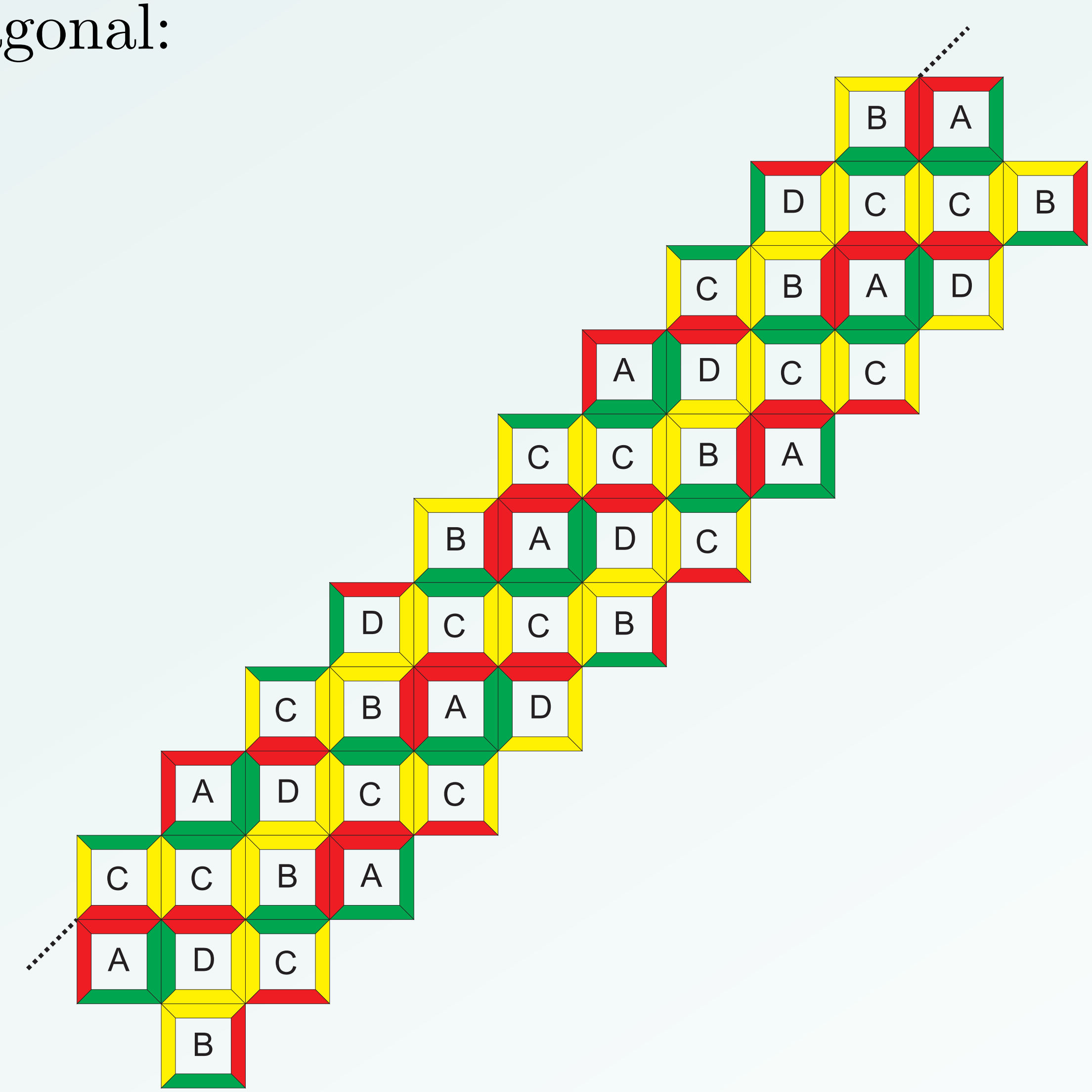

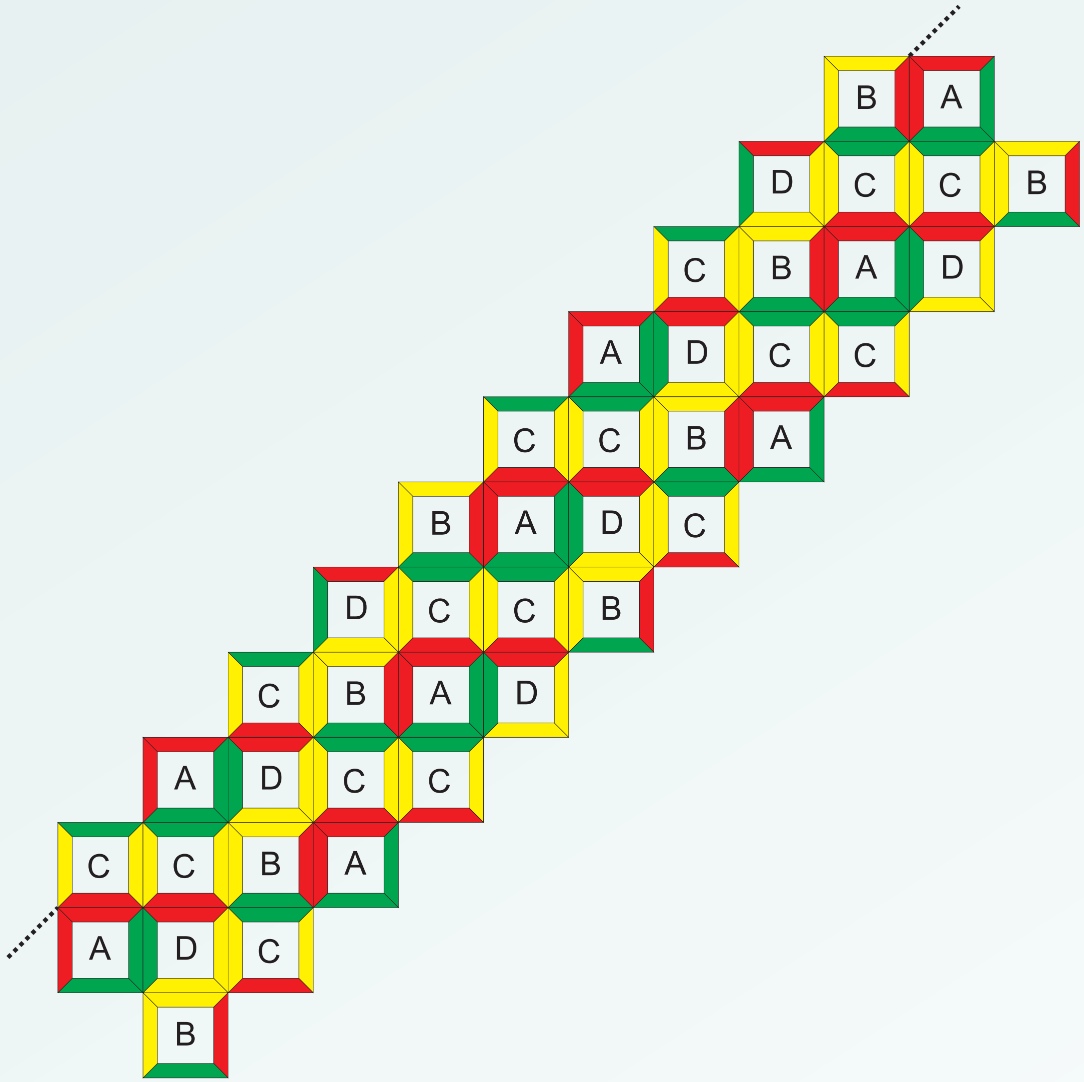

If diagonals are interpreted as configurations of a one-dimensional CA, valid tilings represent space-time diagrams.

To make the CA **reversible**, we may even require that the tile set is **two-way deterministic**: it is deterministic both in NW- and SE-directions.

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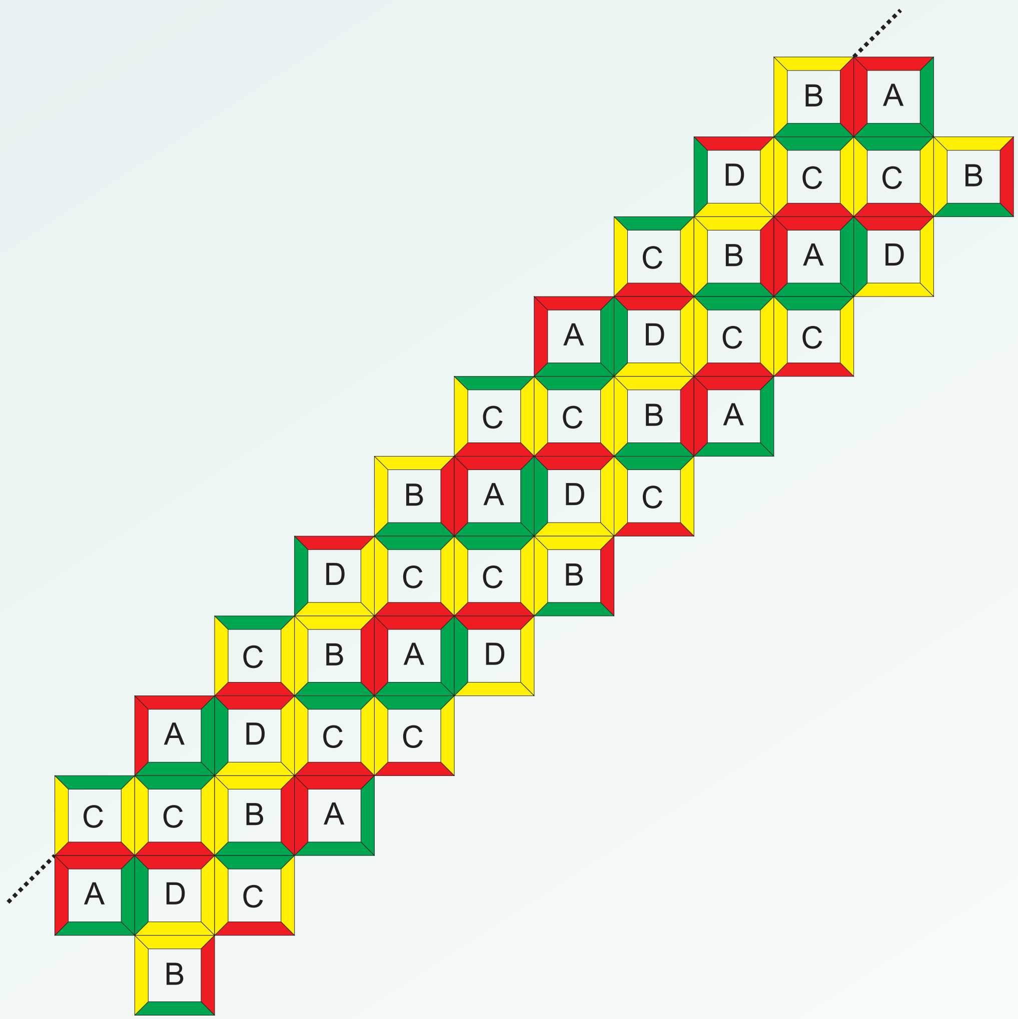

Diagonals determine locally the diagonals below and above.

But are there complex NW-deterministic tile sets? Are they interesting?

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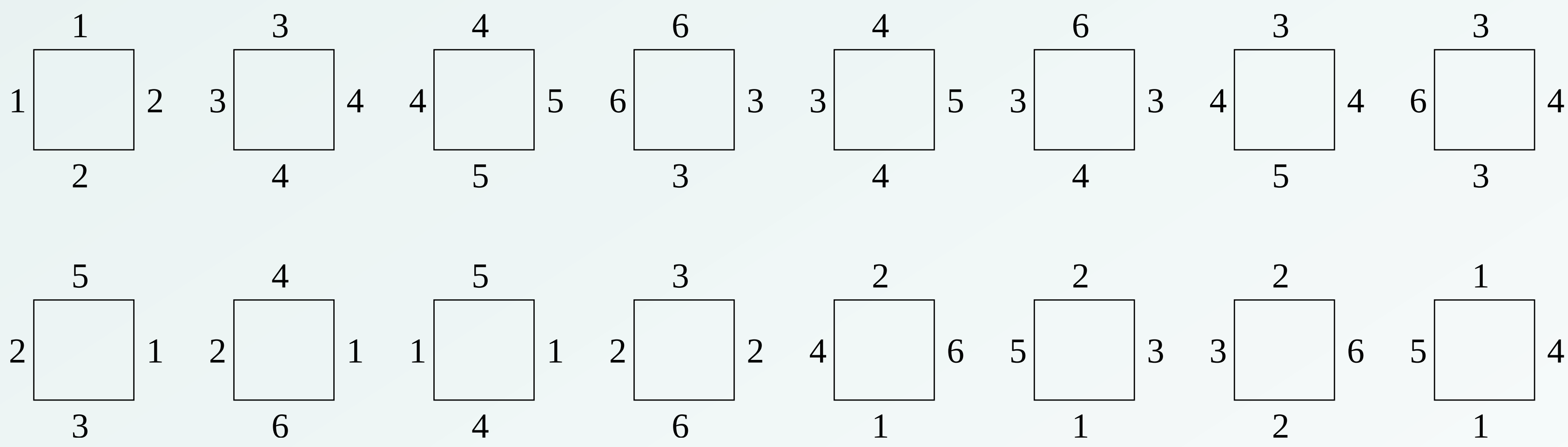
YES!

1. There are **aperiodic** NW-deterministic tiles sets:



Amman's 16 tile aperiodic tile set

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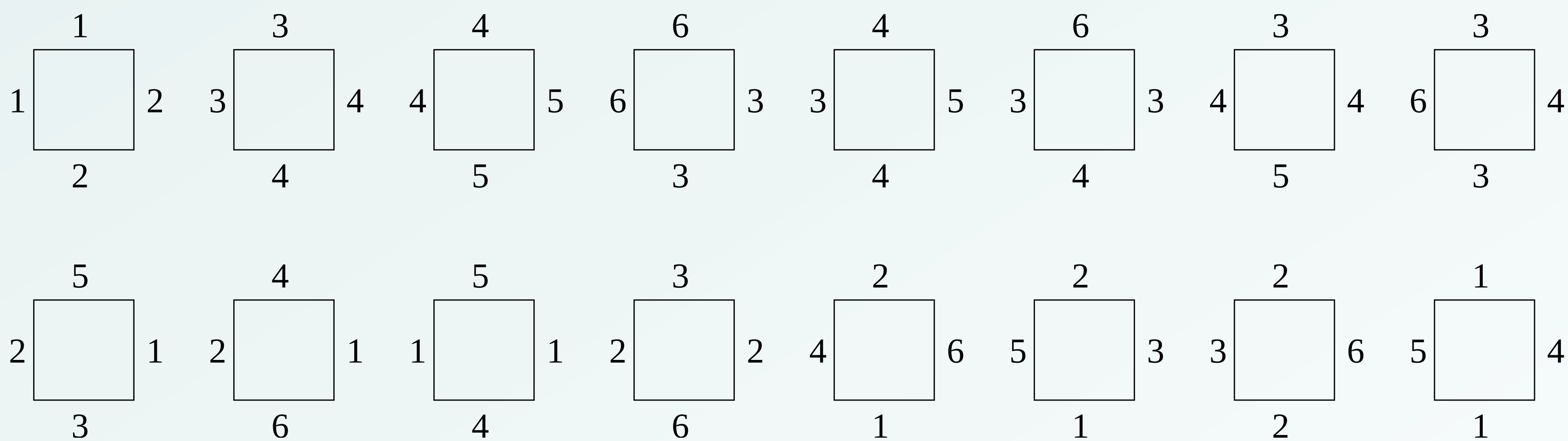

Amman's 16 tile aperiodic tile set

Another example is based on the **Robinson's tiles**.

Ok, there are aperiodic NW-deterministic tile sets. But what about the tiling problem?

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Undecidable!

Theorem. It is undecidable if a given NW-deterministic tile set admits a valid tiling of the plane.

Undecidability in 1D CA

1D nilpotency is undecidable: For any given NW-deterministic tile set T we construct a one-dimensional CA whose

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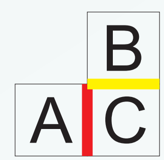
- **state set** is $S = T \cup \{q\}$ where q is a new symbol $q \notin T$,
- **neighborhood** is $(0, 1)$,
- **local rule** $f : S^2 \longrightarrow S$ is defined as follows:
  - $f(A, B) = C$ if the colors match in 

  - $f(A, B) = q$ if $A = q$ or $B = q$ or no matching tile C exists.

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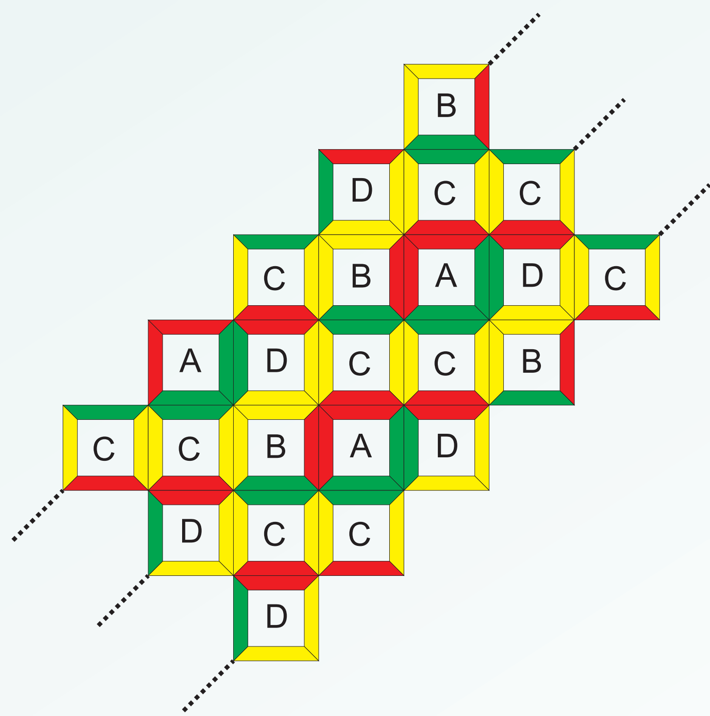
Proof:

$\Longrightarrow$ If T **admits a tiling** c then diagonals of c are configurations that never evolve into the quiescent configuration. So the CA is **not nilpotent**.



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$\Longleftarrow$ If T **does not admit a tiling** then every $n \times n$ square contains a tiling error, for some n. Hence state q is created inside every segment of length n.

Since q spreads, the whole configuration becomes eventually quiescent. The CA is **nilpotent**.

The tiling problem is undecidable for NW-deterministic tile sets, so

Theorem: It is undecidable whether a given one-dimensional CA is nilpotent. $\square$

The tiling problem is undecidable for NW-deterministic tile sets, so

Theorem: It is undecidable whether a given one-dimensional CA is nilpotent. □

If we do the previous construction using an aperiodic set then we have an interesting one-dimensional CA:

- all periodic configurations eventually die, but
- there are non-periodic configurations that never create a quiescent state in any cell.

We have undecidability also on the **two-way deterministic tile sets**.

But even better: Tile set T is **four-way deterministic** if it is deterministic in all four directions NW, SW, SE and NE (cf. bi-reversible automata from yesterday).

Theorem (Lukkarila 2008) The tiling problem is undecidable among 4-way deterministic tile sets.

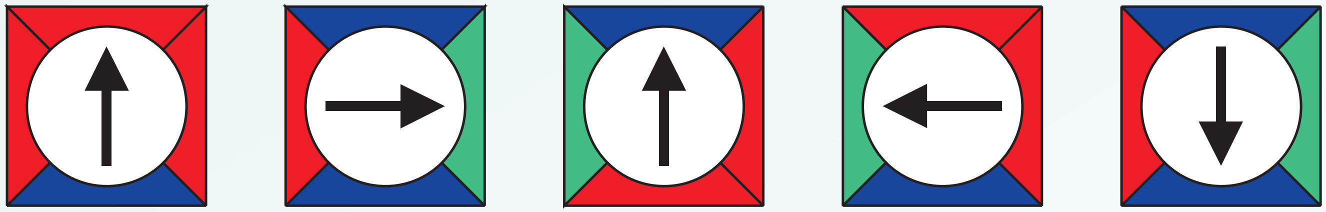
This result provides some undecidability results for dynamics of **reversible** one-dimensional CA.

Back to 2D CA: Decision problems to determine if a given 2D CA is **reversible** or **surjective**.

SNAKES®

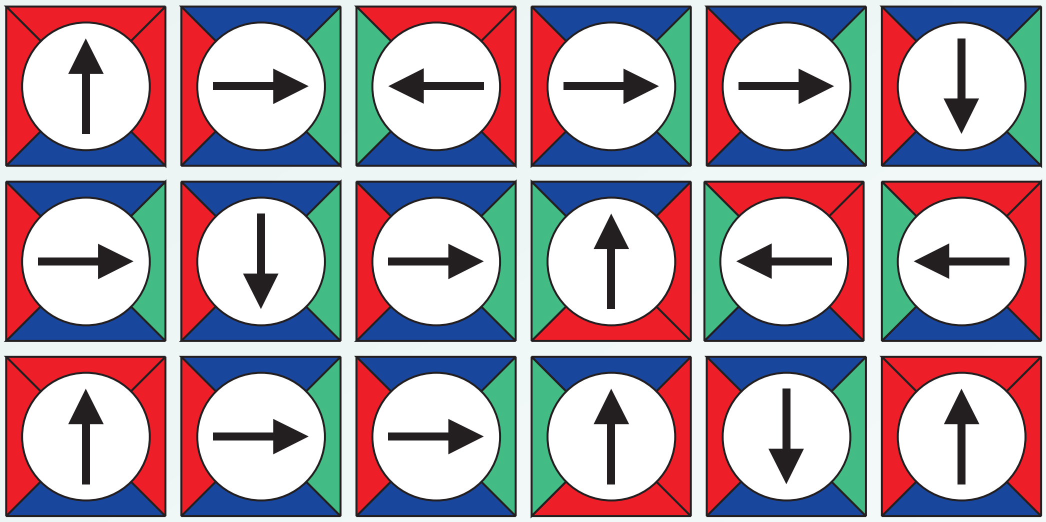
SNAKES is a tile set with some interesting (and useful) properties.

In addition to colored edges, these tiles also have an arrow printed on them. The arrow is horizontal or vertical and it points to one of the four neighbors of the tile:



Such tiles with arrows are called **directed tiles**.

Given a configuration (valid tiling or not!) and a starting position, the arrows specify a path on the plane. Each position is followed by the neighboring position indicated by the arrow of the tile:



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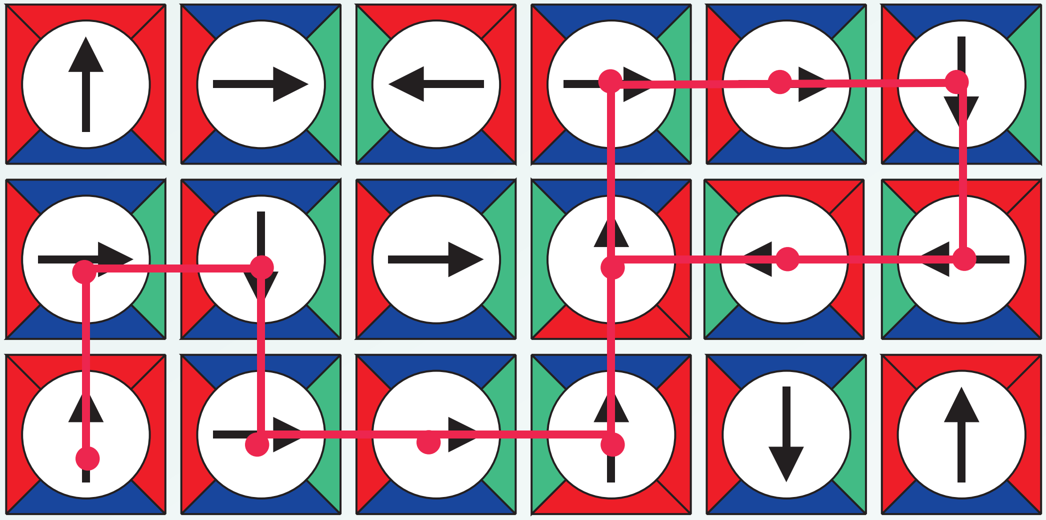
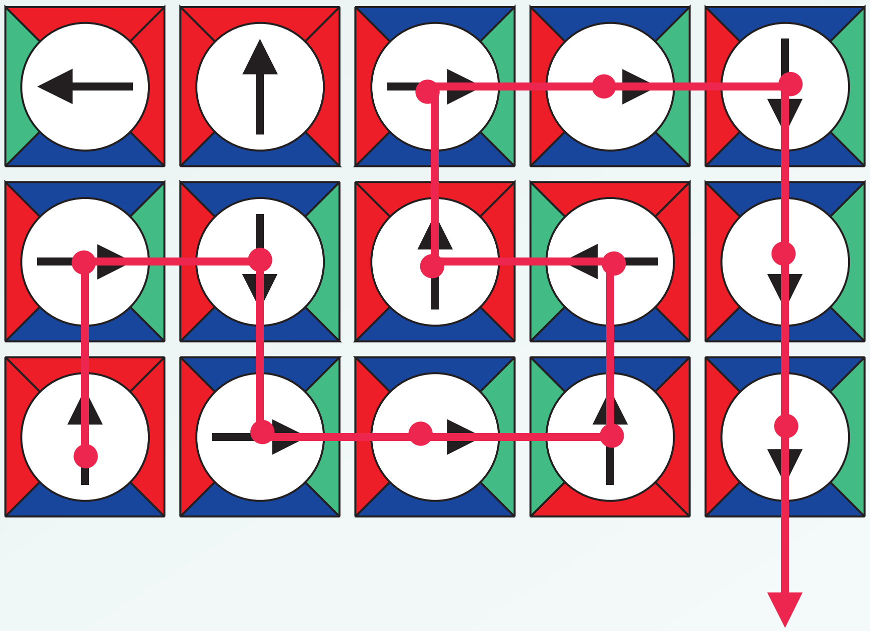


The path may enter a loop...

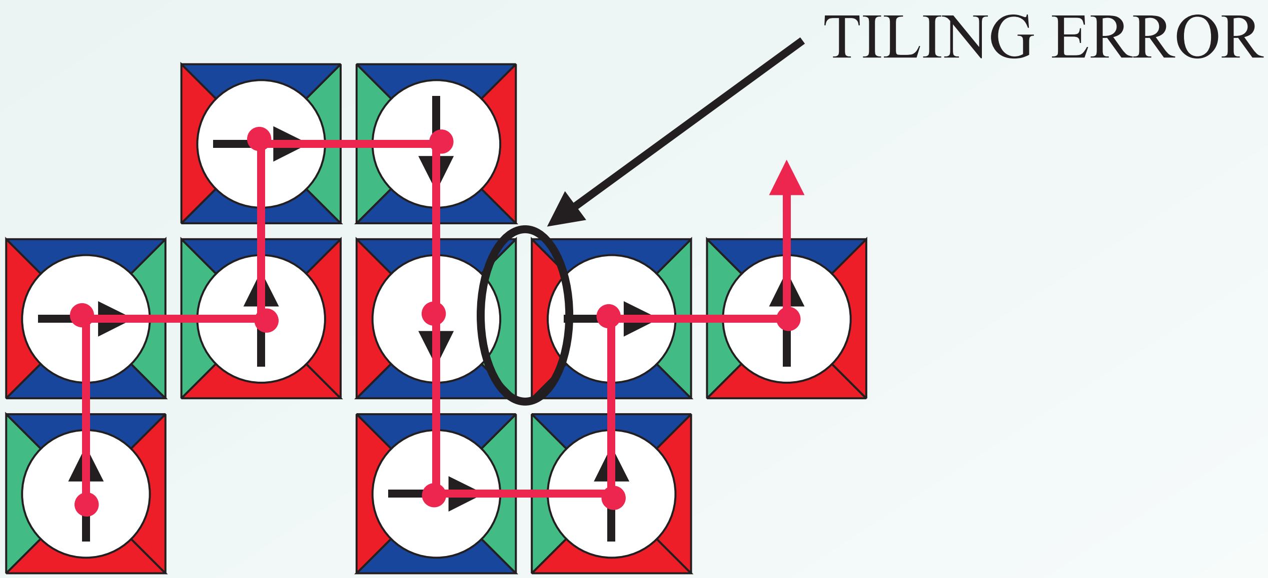
Given a configuration (valid tiling or not!) and a starting position, the arrows specify a path on the plane. Each position is followed by the neighboring position indicated by the arrow of the tile:



. . . or the path may be infinite and never return to a tile visited before.

The directed tile set SNAKES has the following property: On any configuration (valid tiling or not) and on any path that follows the arrows one of the following two things happens:

(1) Either there is a tiling error at a tile of the path,



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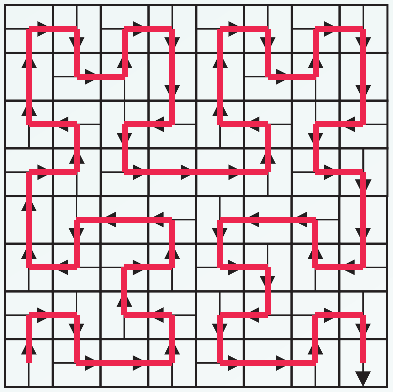
(1) Either there is a tiling error at a tile of the path,

(2) or the path is a plane-filling path, that is, for every positive integer n there exists an $n \times n$ square all of whose positions are visited by the path.



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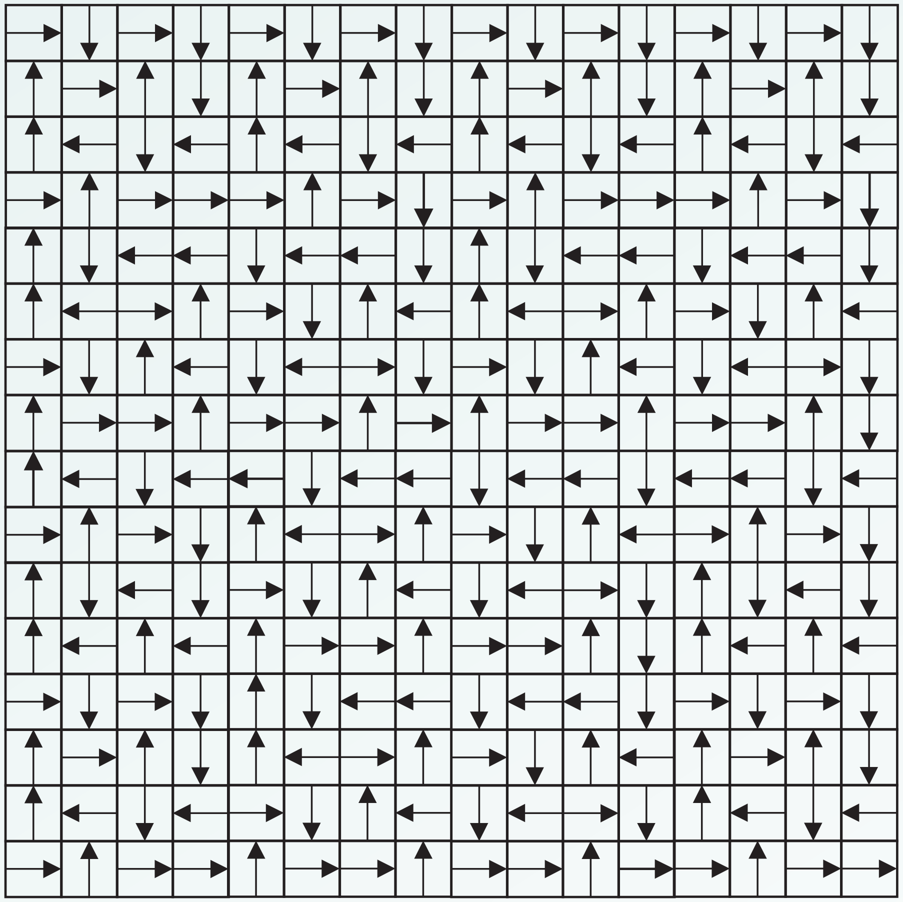
Note that the tiling may be invalid outside path P, yet the path is forced to snake through larger and larger squares.

SNAKES also has the property that it admits a valid tiling.

The paths that SNAKES forces when no tiling error is encountered have the shape of the well known plane-filling Hilbert-curve



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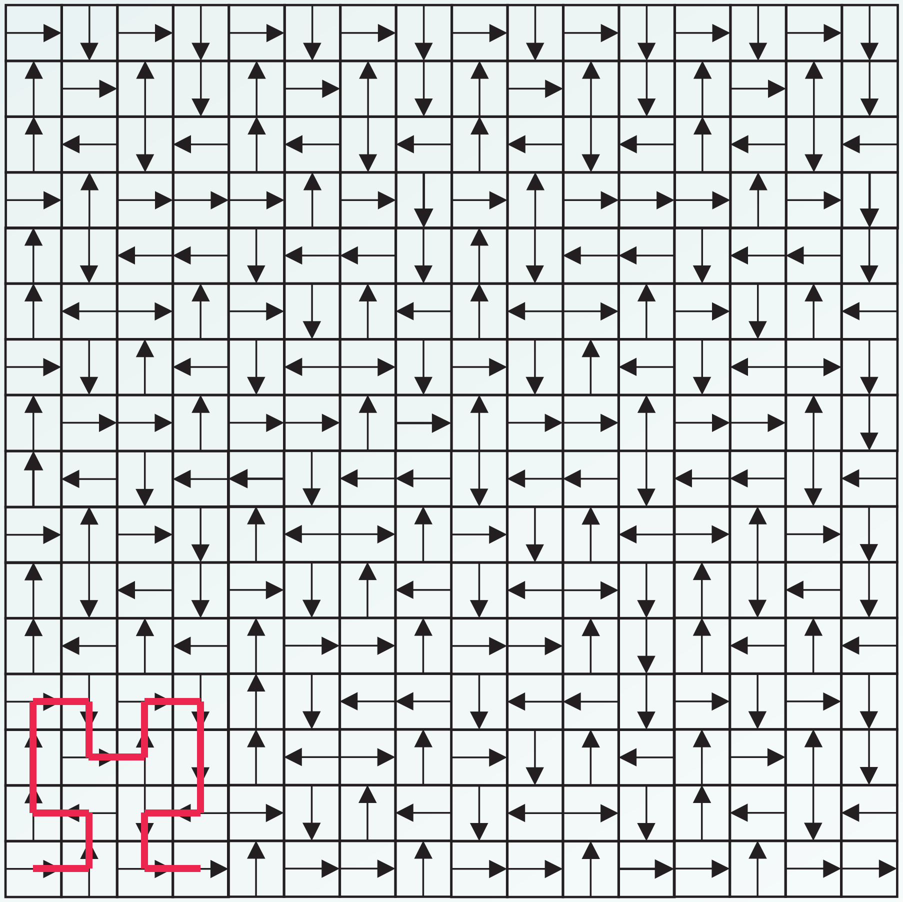

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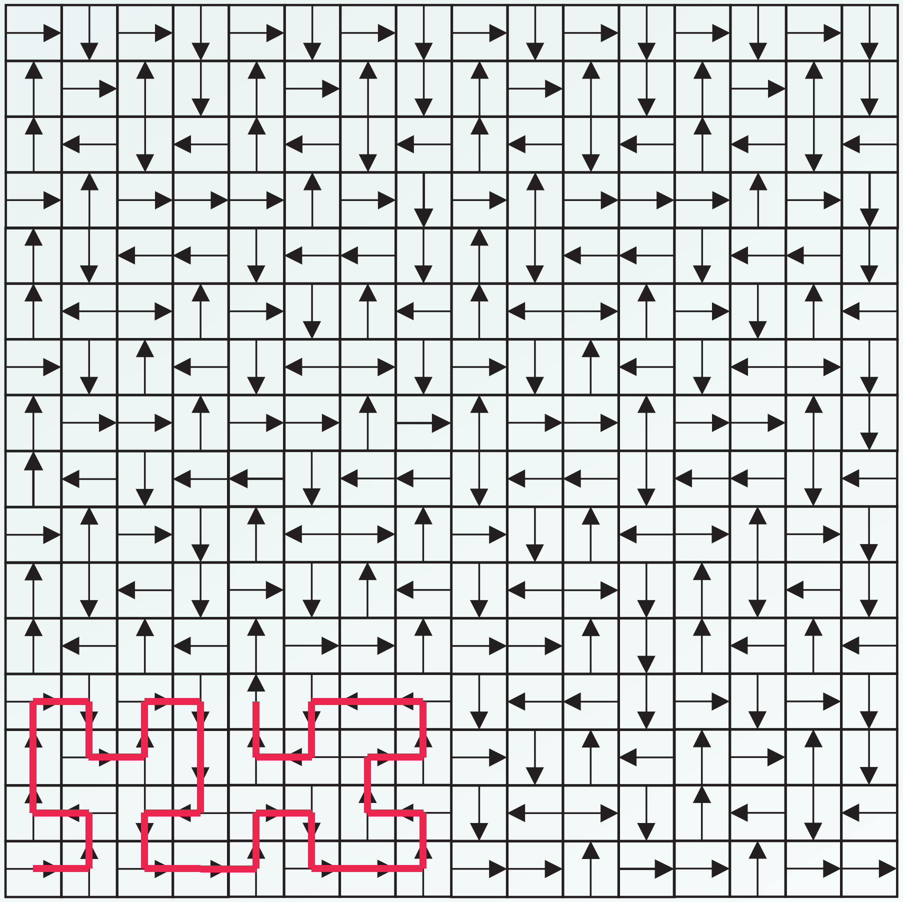

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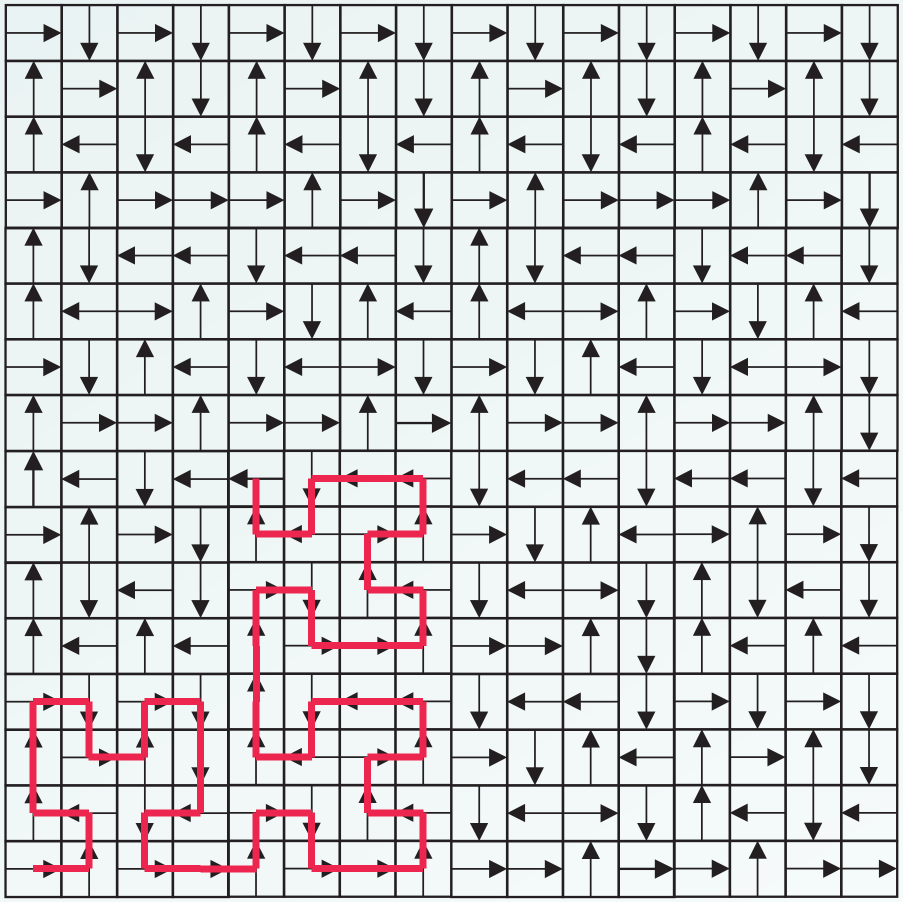

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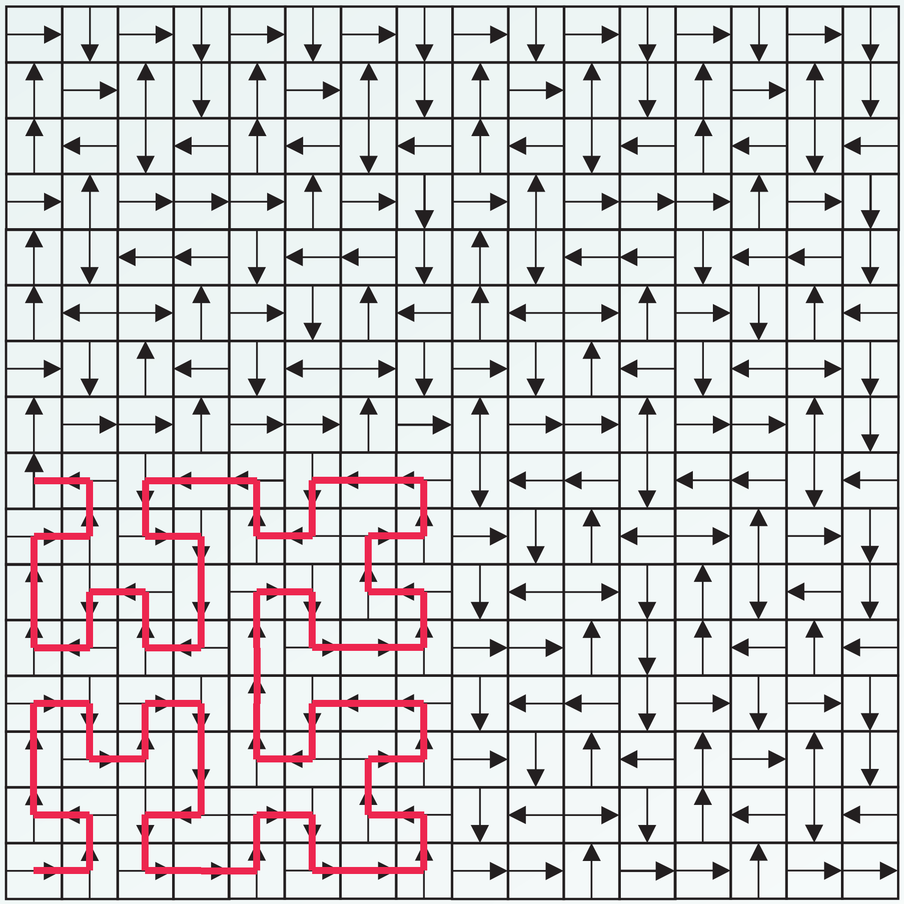

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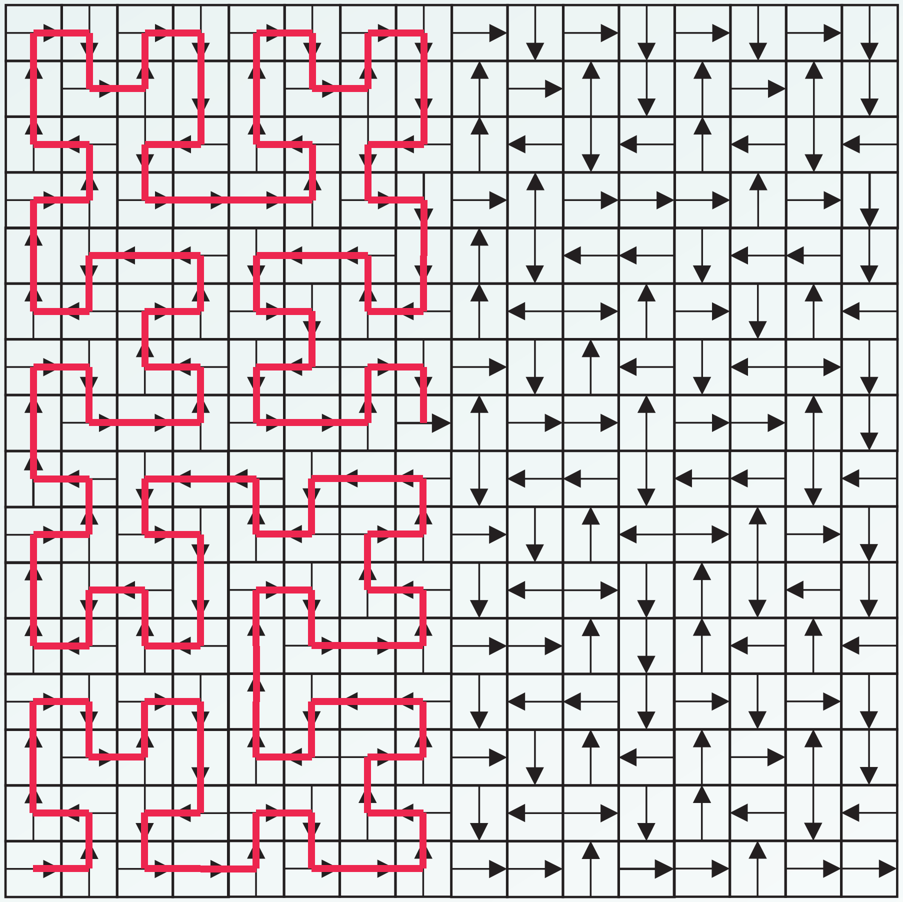

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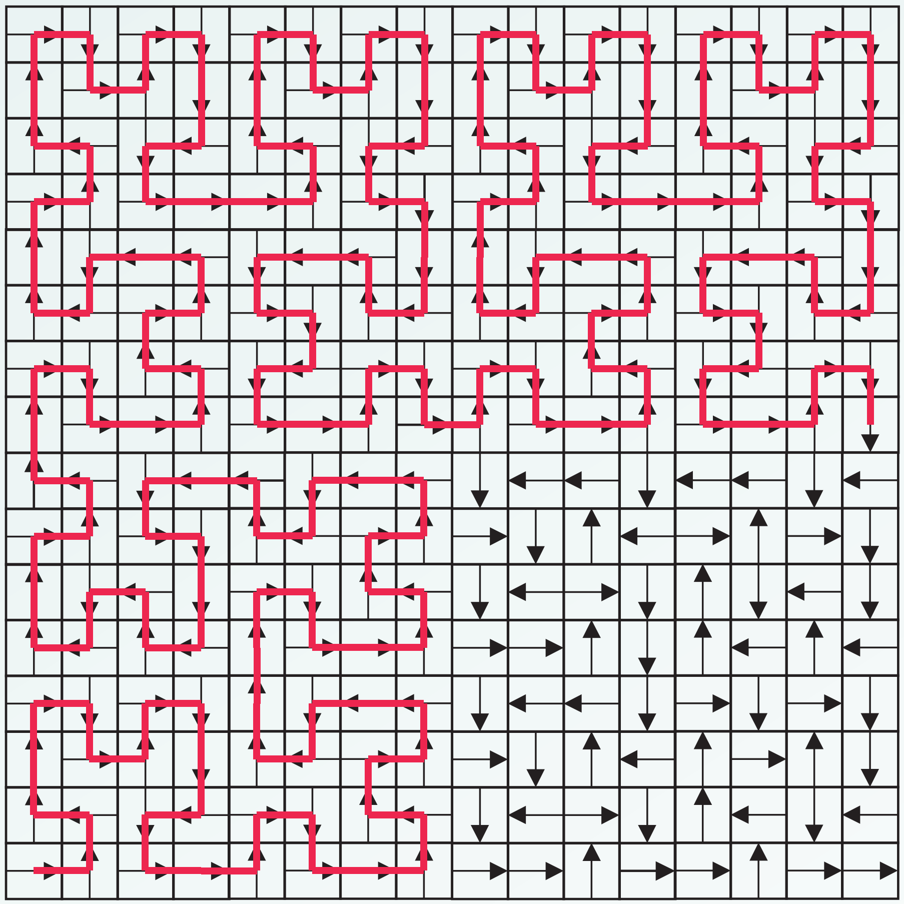
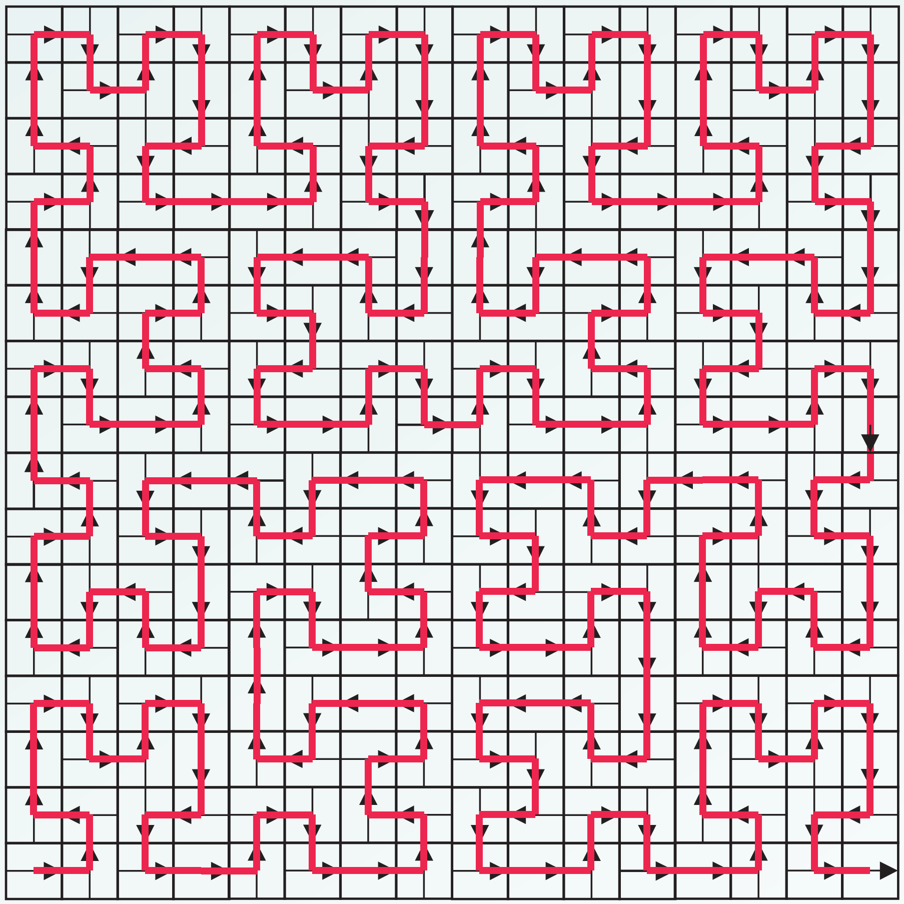


The paths that Snakes forces when no tiling error is encountered have the shape of the well known plane-filling Hilbert-curve



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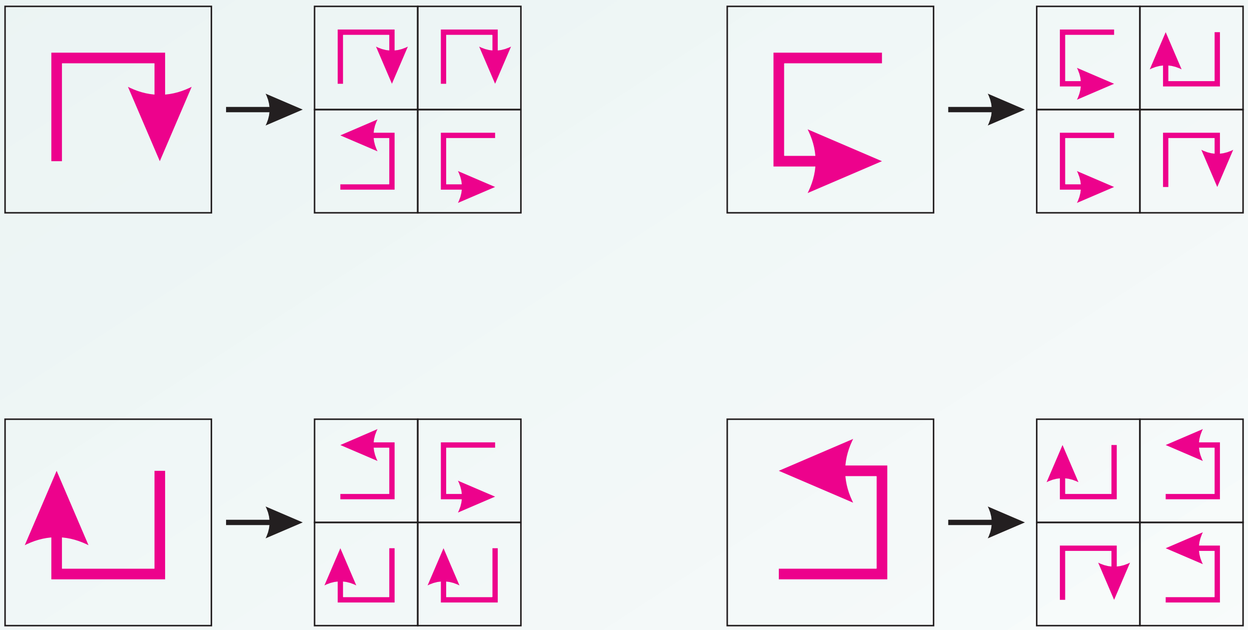
Hilbert-curve comes in four orientations, generated by substitutions



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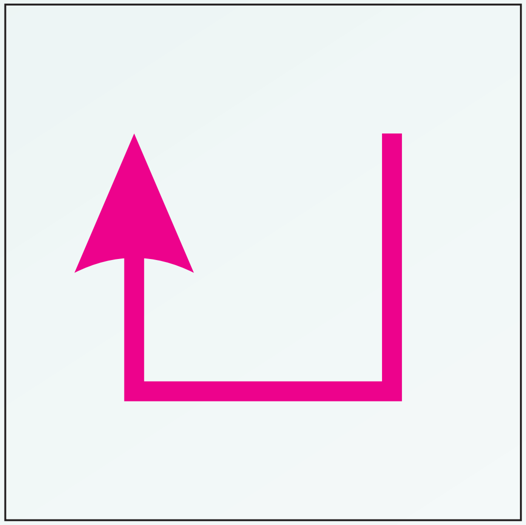

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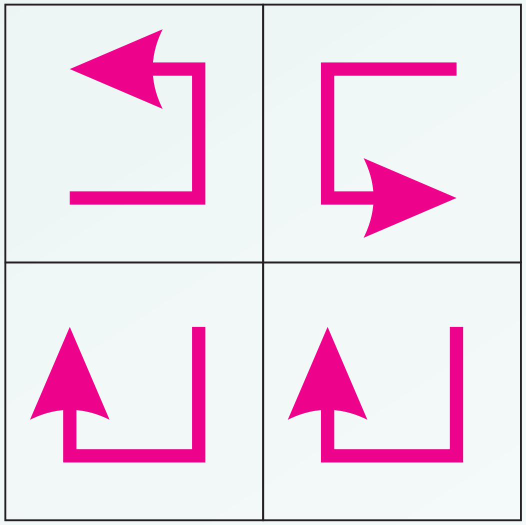

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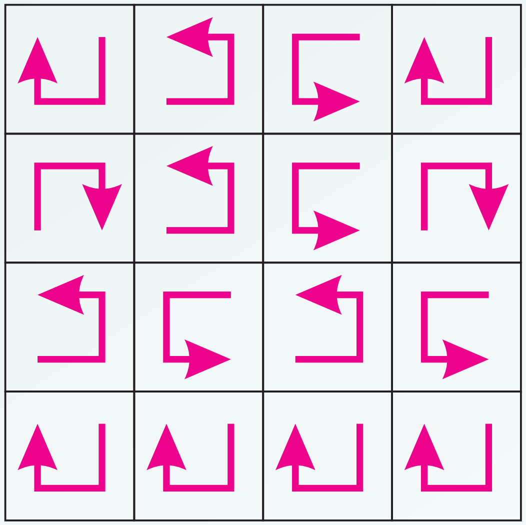
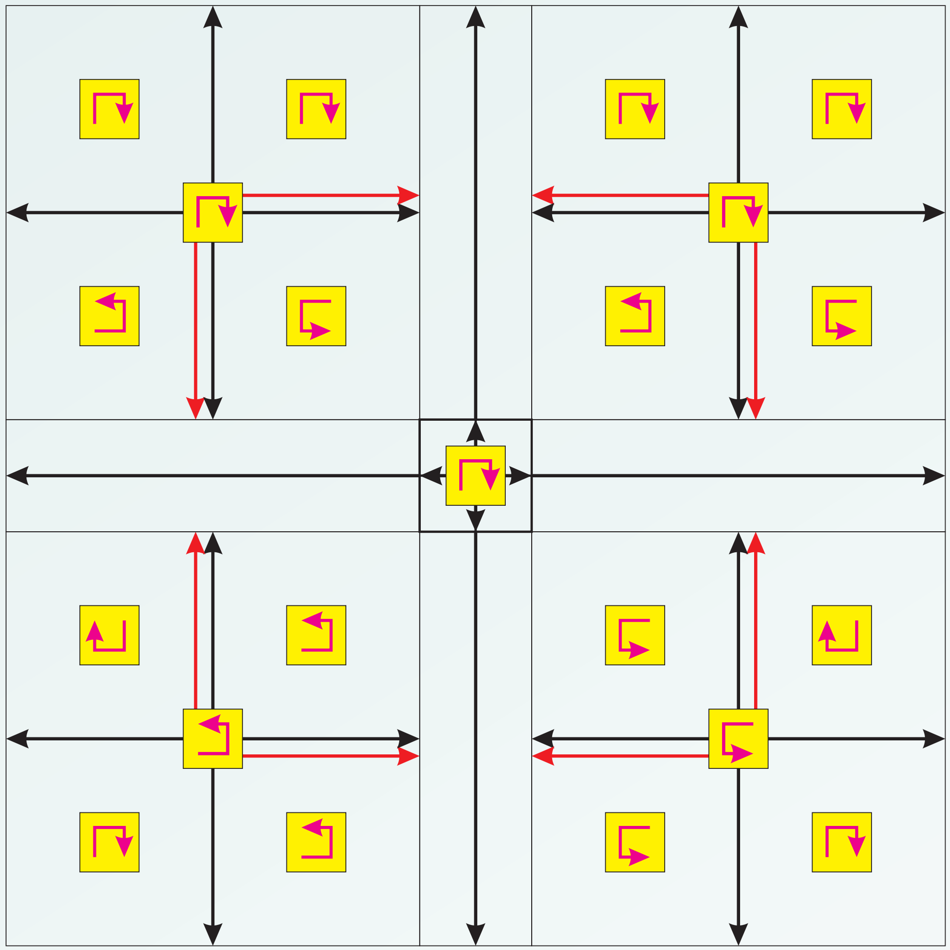




The recursive structures of the Hilbert-curve and the Robinson's tiling are consistent.

Applications of Snakes

First application of Snakes: An example of a two-dimensional CA that is injective on periodic configurations but is not injective on all configurations.

The **Snake XOR** CA confirms that in 2D

$$G \text{ injective} \quad \not\Leftarrow \quad G_P \text{ injective.}$$

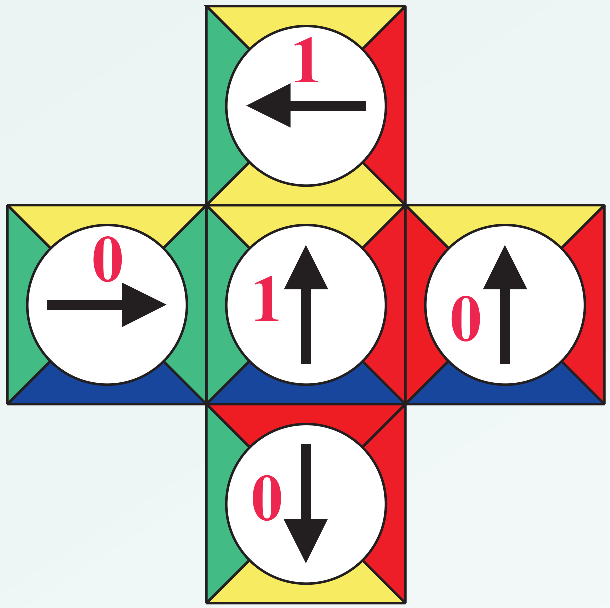
The state set of the CA is

$$S = \textsc{Snakes} \times \{0, 1\}.$$

(Each snake tile is attached a red bit.)

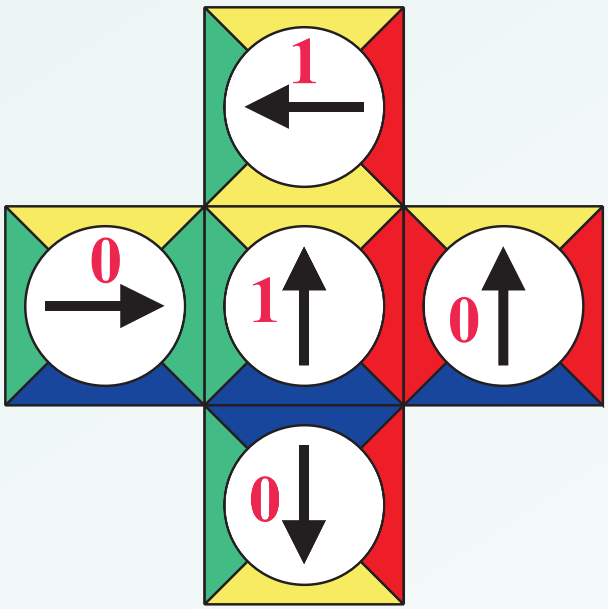
The local rule checks whether the tiling is valid at the cell:

- If there is a tiling error, no change in the state.



The local rule checks whether the tiling is valid at the cell:

- If there is a tiling error, no change in the state.
- If the tiling is valid, the cell is **active**: the bit of the neighbor next on the path is XOR'ed to the bit of the cell.



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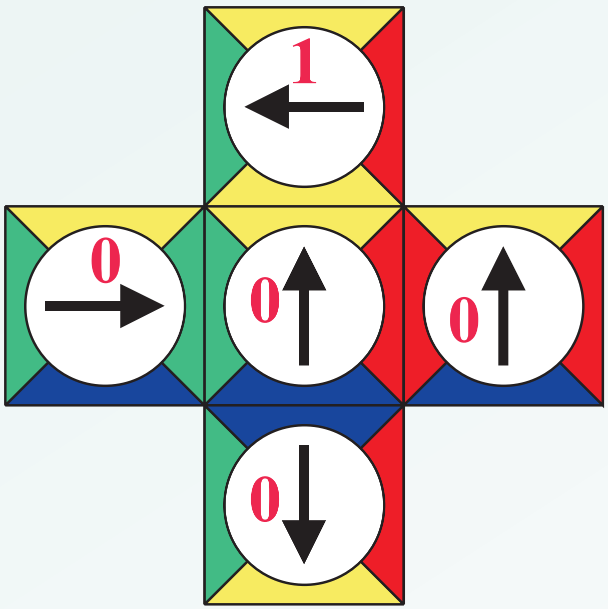

Snake XOR is not injective:

The following two configurations have the same successor: The SNAKES tilings of the configurations form the same valid tiling of the plane. In one of the configurations all bits are set to 0, and in the other configuration all bits are 1.

All cells are active because the tilings are correct. This means that all bits in both configurations become 0. So the two configurations become identical. The CA is not injective.

Snake XOR is injective on periodic configurations:

Suppose there are different periodic configurations c and d with the same successor. Since only bits may change, c and d must have identical SNAKES tiles everywhere. So they must have different bits 0 and 1 in some position $\vec{p}_1 \in \mathbb{Z}^2$.

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Because c and d have identical successors:

- The cell in position $\vec{p}_1$ must be active, that is, the SNAKES tiling is valid in position $\vec{p}_1$.
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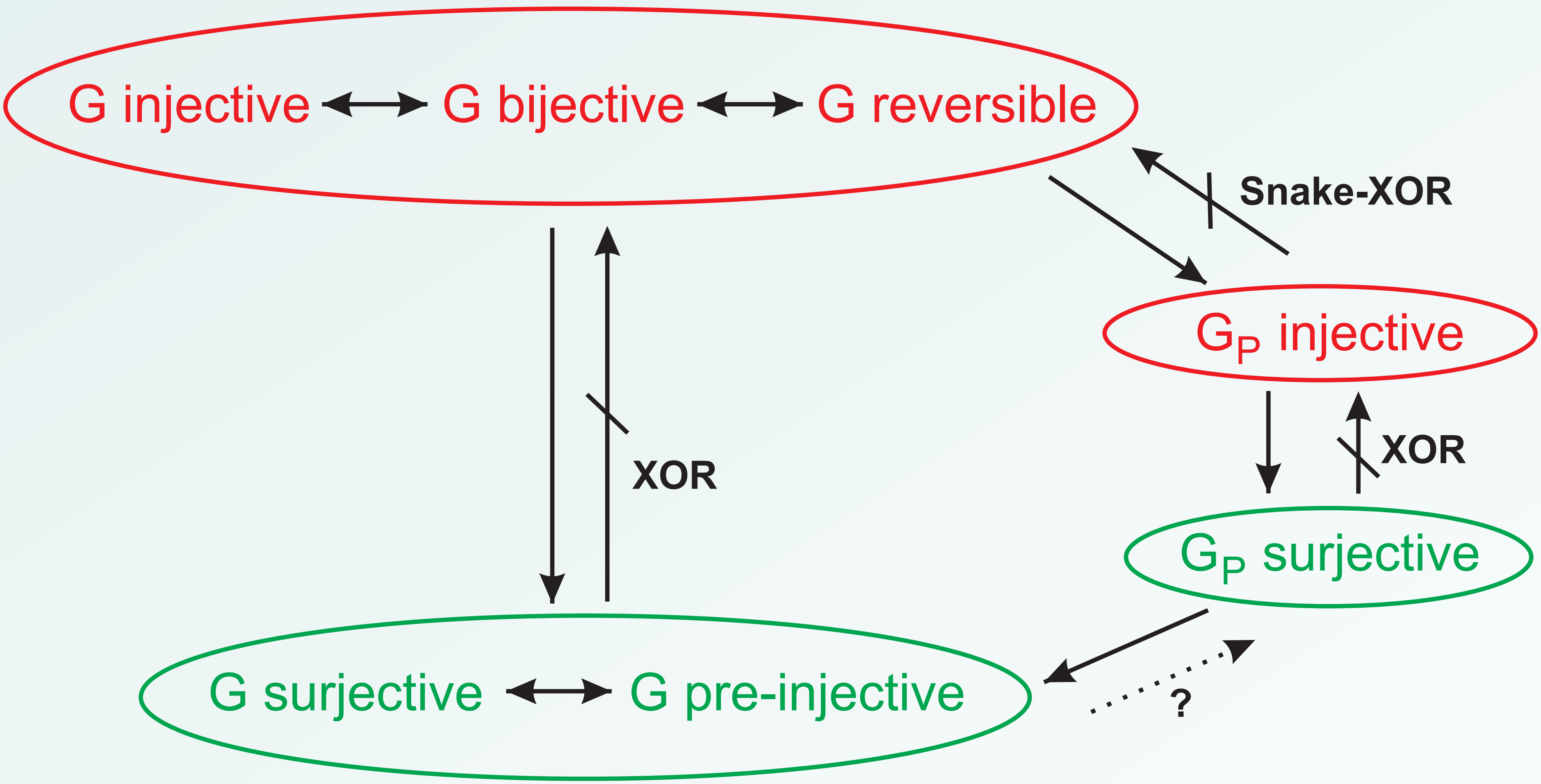
Hence we can repeat the reasoning in position $\vec{p}_2$.

The same reasoning can be repeated over and over again. The positions $\vec{p}_1, \vec{p}_2, \vec{p}_3, \ldots$ form a path that follows the arrows on the tiles. There is no tiling error at any tile on this path.

But this contradicts the fact that the plane filling property of SNAKES guarantees that on periodic configuration every path encounters a tiling error. $\square$

In 2D


G injective
G bijective
G reversible
Snake-XOR
Gₚ injective
XOR
XOR
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?

Snake XOR also refutes an earlier conjecture that all 2D CA have either infinite entropy or zero entropy. It has finite but non-zero topological entropy (T.Meyerovitch).

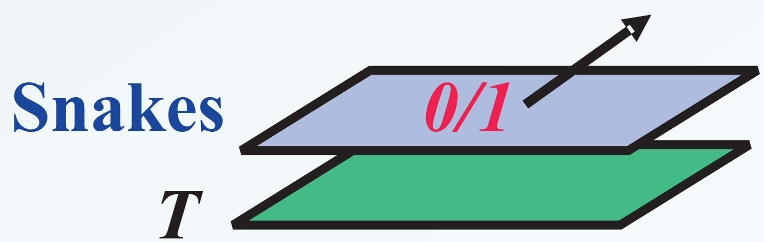
Second application of Snakes: It is undecidable to determine if a given two-dimensional CA is reversible.

Second application of SNAKES: It is undecidable to determine if a given two-dimensional CA is reversible.

The proof is a reduction from the tiling problem, using the tile set SNAKES.

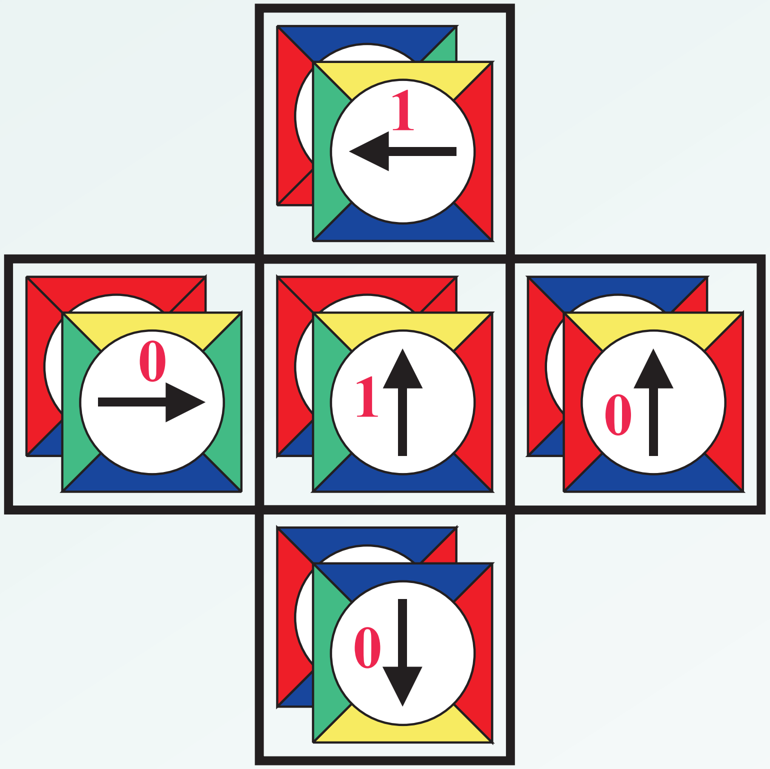
For any given tile set T we construct a CA with the state set

$$S = T \times \text{SNAKES} \times \{0, 1\}.$$



The local rule is analogous to **Snake XOR** with the difference that the correctness of the tiling is checked in both tile layers:

- If there is a tiling error then the cell is inactive.



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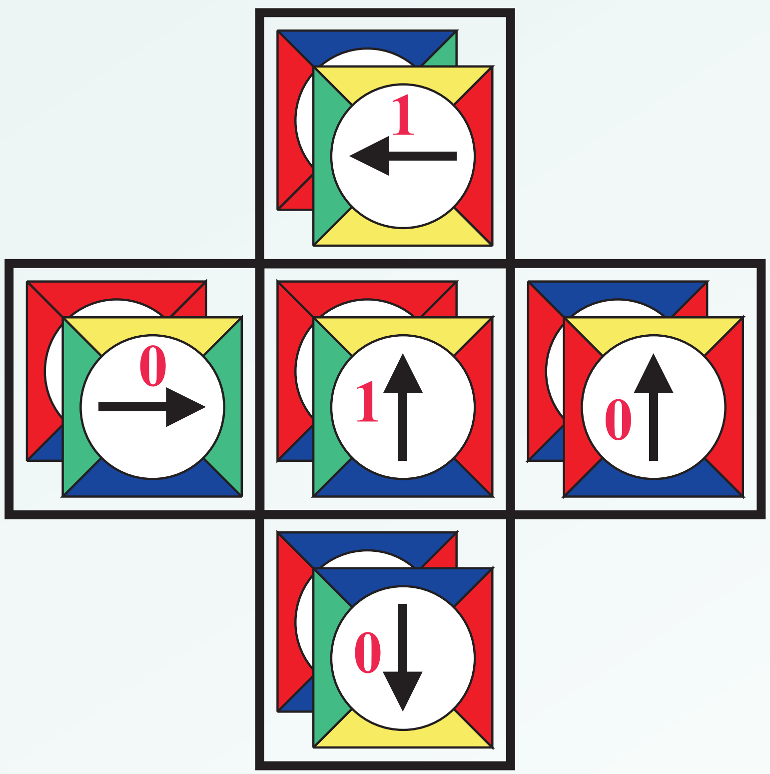

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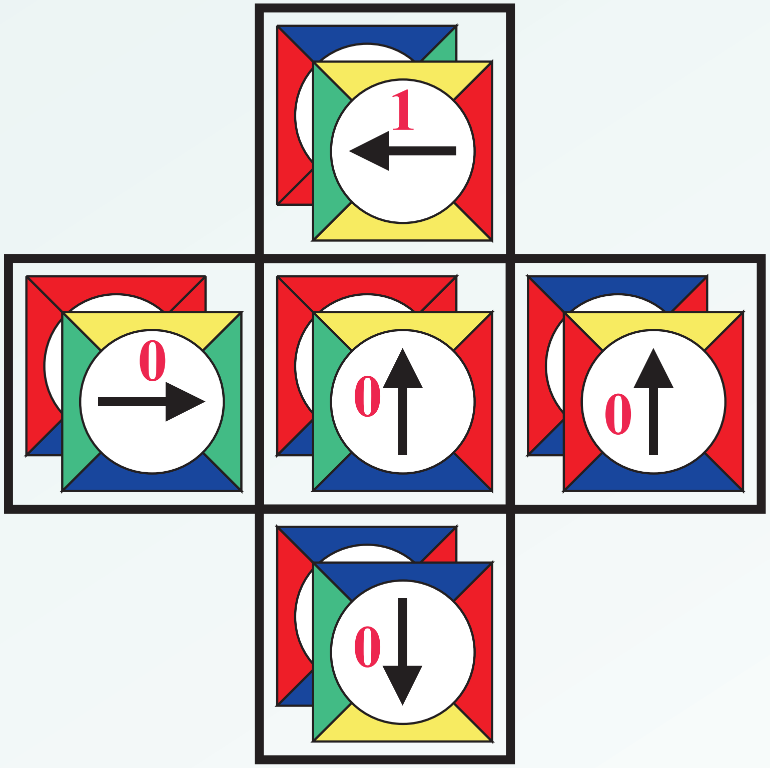

We can reason exactly as with **Snake XOR**:

T admits a tiling $\iff$ the CA is not reversible

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T admits a tiling $\iff$ the CA is not reversible

($\Longrightarrow$) Suppose a valid tiling exists.

Make two configurations c_0 and c_1 whose SNAKES and the T layers form the same valid tilings of the plane. In c_0 all bits are 0, in c_1 all bits are 1.

All cells are active because the tilings are correct. So all bits in both configurations become 0, hence $G(c_0) = G(c_1)$. The CA is not injective.

($\Longleftarrow$) Conversely, assume that the CA is not injective. Let c and d be two different configurations with the same successor. Since only bits may change, c and d must have identical SNAKES and T layers. So they must have different bits 0 and 1 in some position $\vec{p}_1 \in \mathbb{Z}^2$.

($\Longleftarrow$) Conversely, assume that the CA is not injective. Let c and d be two different configurations with the same successor. Since only bits may change, c and d must have identical SNAKES and T layers. So they must have different bits 0 and 1 in some position $\vec{p}_1 \in \mathbb{Z}^2$.

Because c and d have identical successors:

- The cell in position $\vec{p}_1$ must be active, that is, the SNAKES and T tilings are both valid in position $\vec{p}_1$.
- The bits stored in the next position $\vec{p}_2$ (indicated by the direction) are different in c and d.

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The same reasoning can be repeated over and over again. The positions $\vec{p}_1, \vec{p}_2, \vec{p}_3, \ldots$ form a path that follows the arrows on the tiles. There is no tiling error at any tile on this path so the special property of SNAKES forces the path to cover arbitrarily large squares.

Hence T admits tilings of arbitrarily large squares, and consequently a tiling of the infinite plane. $\square$

Theorem: It is undecidable whether a given two-dimensional CA is injective.

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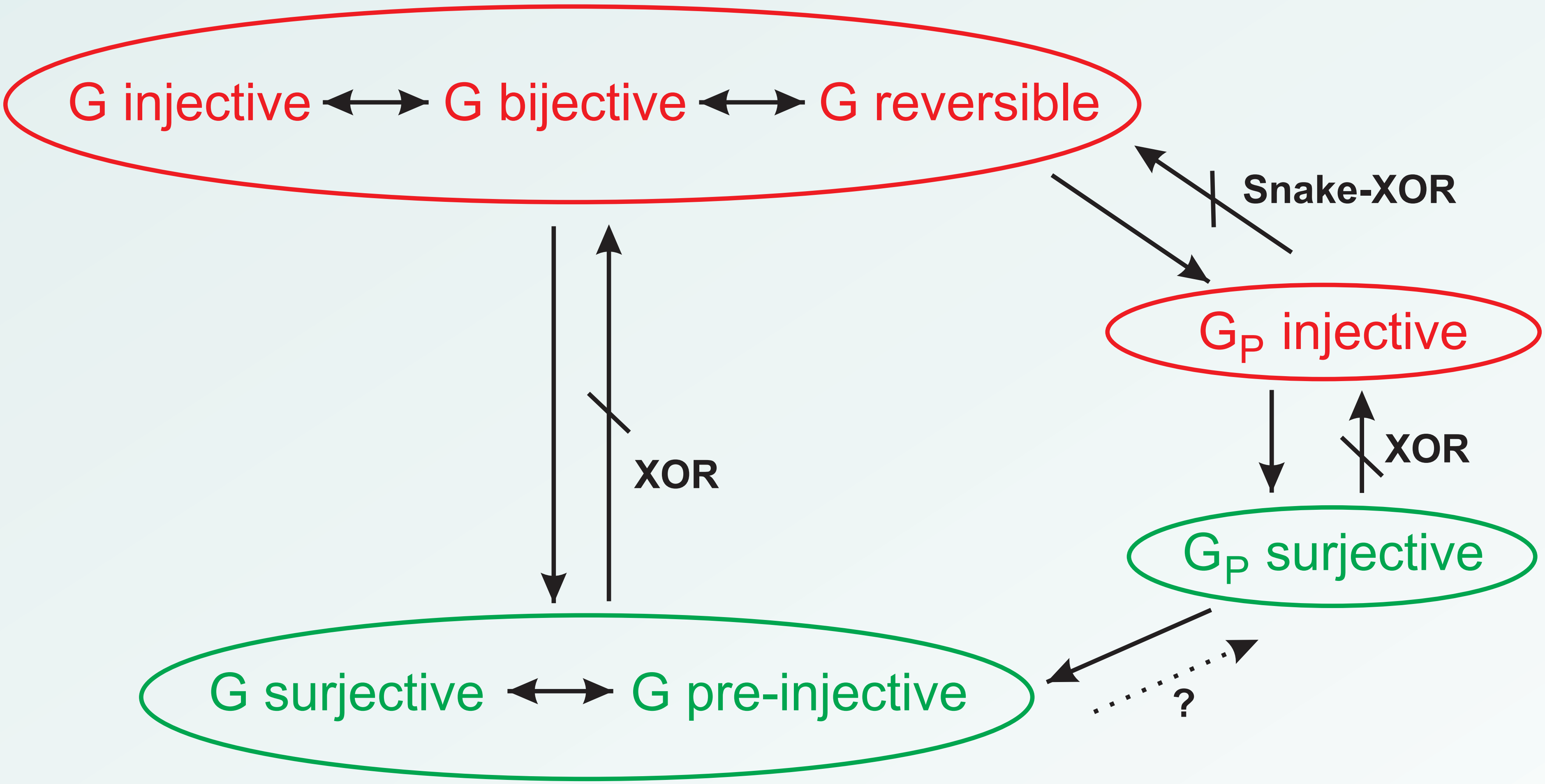
An analogous (but simpler!) construction can be made for the surjectivity problem, based on the fact surjectivity is equivalent to pre-injectivity:

Theorem: It is undecidable whether a given two-dimensional CA is surjective.



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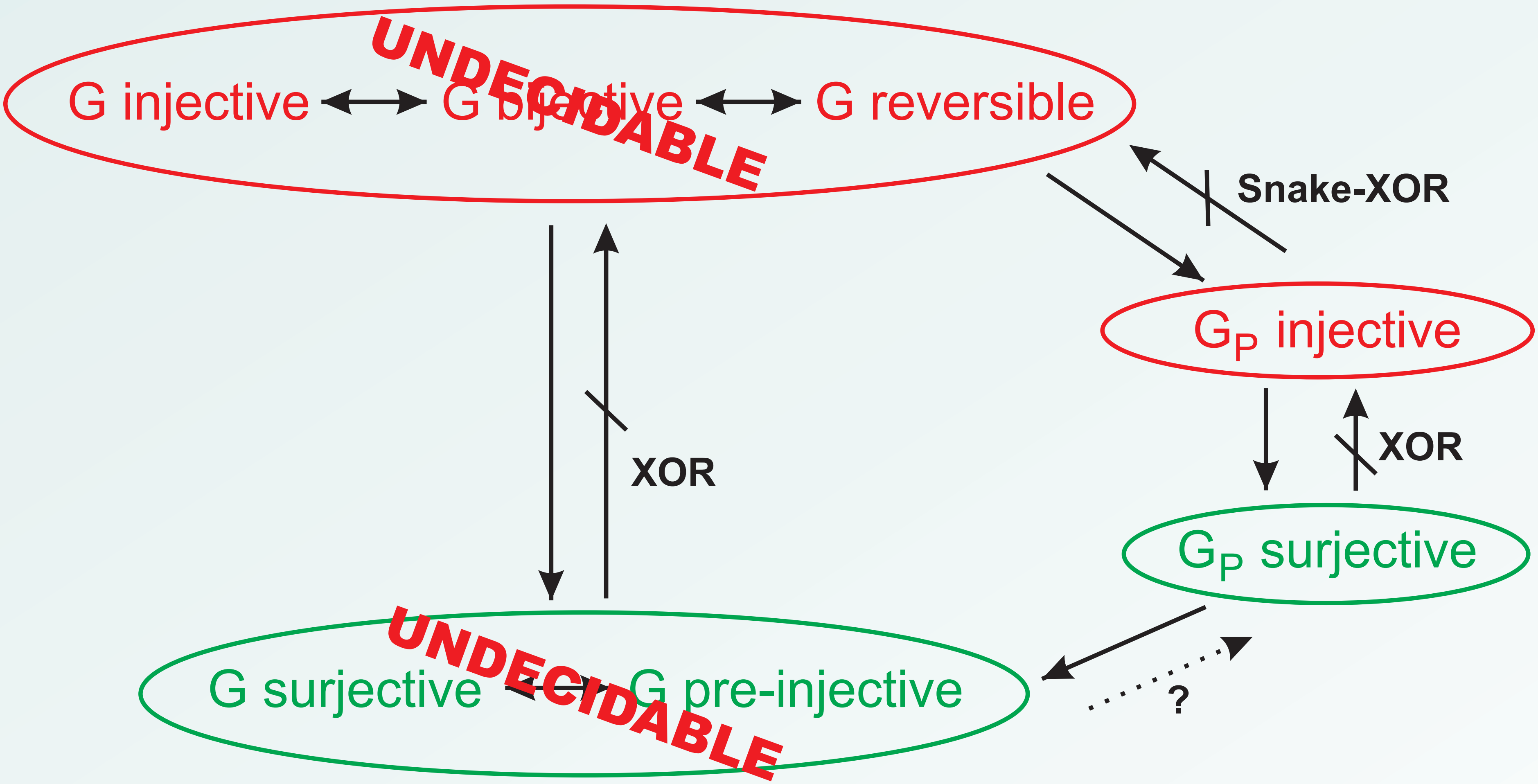
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UNDECIDABLE
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XOR
XOR
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G surjective
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?

Both problems are semi-decidable in one direction:

Injectivity is semi-decidable: Enumerate all CA G one-by one and check if G is the inverse of the given CA. Halt once (if ever) the inverse is found.

Non-surjectivity is semi-decidable: Enumerate all finite patterns one-by-one and halt once (if ever) an orphan is found.

Undecidability of injectivity implies the following:

There are some reversible CA that use von Neumann neighborhood but whose inverse automata use a very large neighborhood: There can be no computable upper bound on the extend of this inverse neighborhood.

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There are some reversible CA that use von Neumann neighborhood but whose inverse automata use a very large neighborhood: There can be no computable upper bound on the extend of this inverse neighborhood.

Topological arguments $\Longrightarrow$ A finite neighborhood is enough to determine the previous state of a cell.

Computation theory $\Longrightarrow$ This neighborhood may be extremely large.

Undecidability of surjectivity implies the following:

There are non-surjective CA whose smallest orphan is very large: There can be no computable upper bound on the extend of the smallest orphan.

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There are non-surjective CA whose smallest orphan is very large: There can be no computable upper bound on the extend of the smallest orphan.

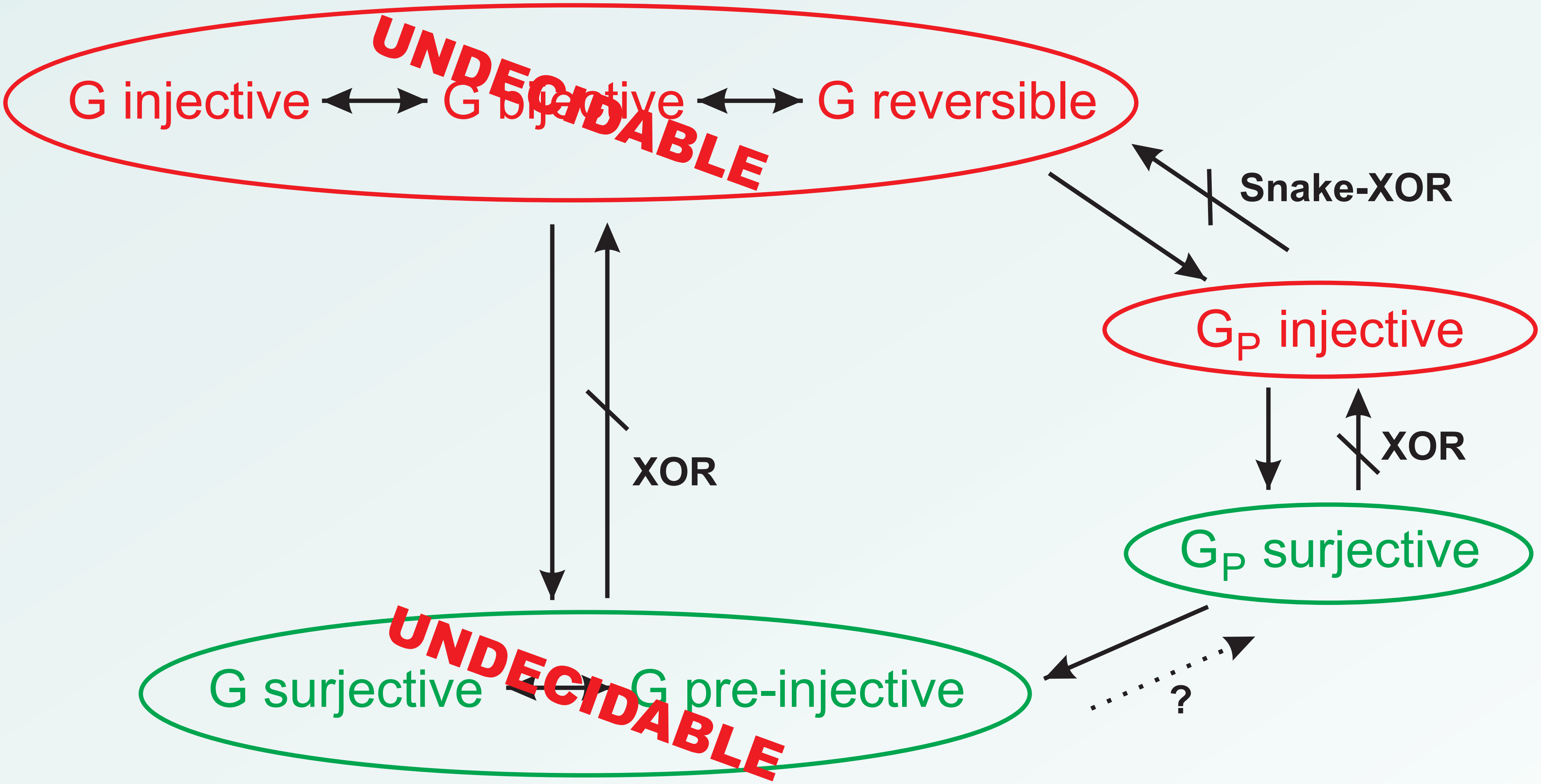
So while the smallest known orphan for Game-Of-Life is pretty big (88 cells), this pales in comparison with some other CA.

The undecidability proofs for reversibility and surjectivity can be merged into

Theorem: The classes of

- Reversible 2D CA
- Non-surjective 2D CA

are recursively inseparable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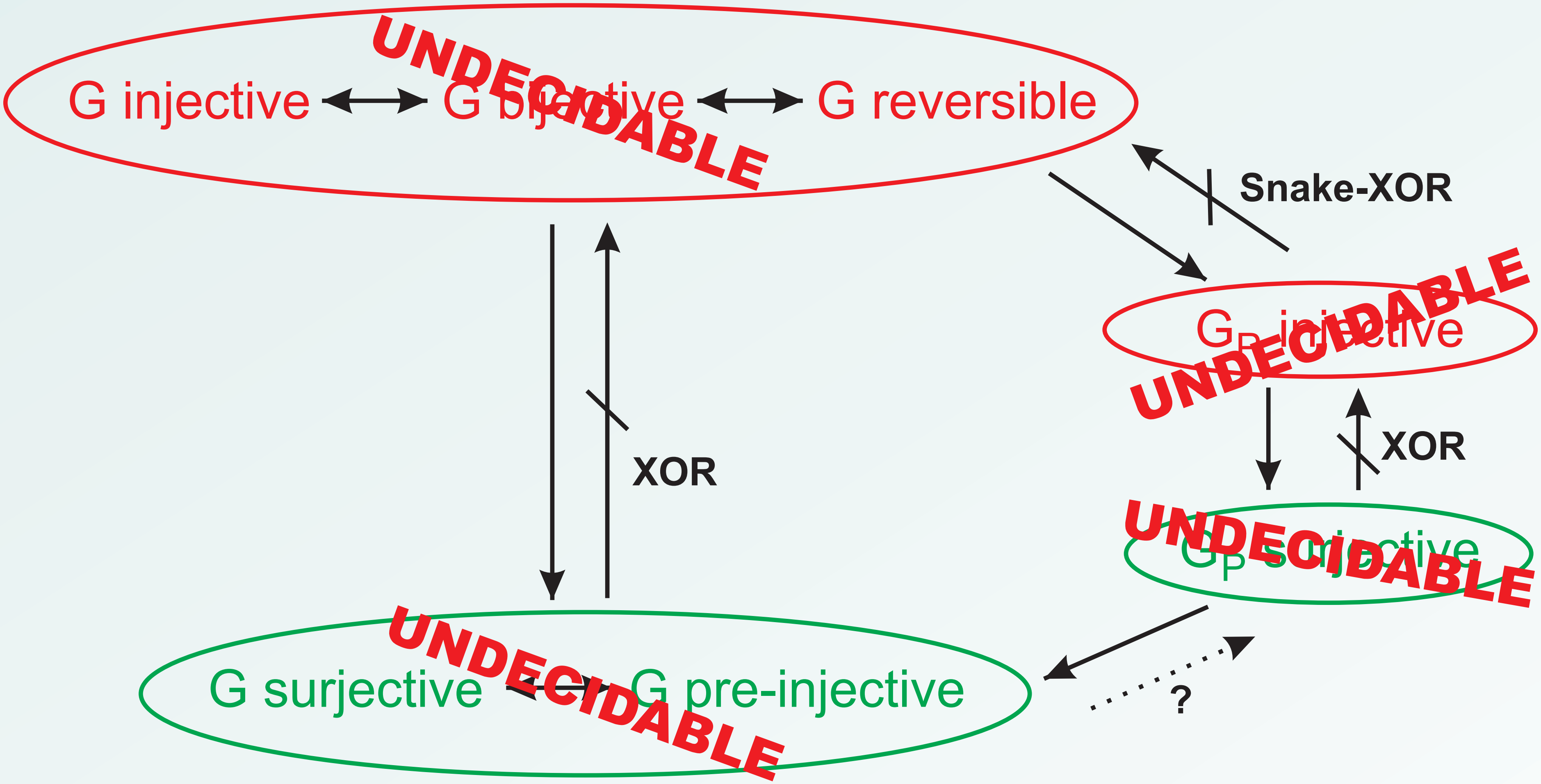

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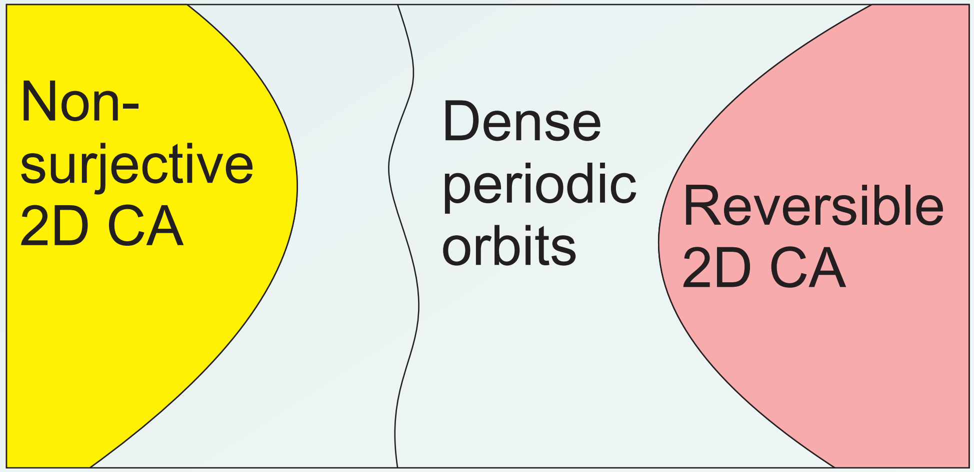

Example: It is conjectured that all surjective CA have dense periodic orbits. This is surely the case for all reversible CA, but not the case for any non-surjective CA.

We have no idea on the solution of this long standing open problem but, in any case, we know now that it is undecidable if a given 2D CA has dense periodic orbits.

Thank You

for your

attention