Substitution cut-and-project tilings with n-fold rotational symmetry

Victor Hubert Lutfalla

Turun yliopisto
University of Turku

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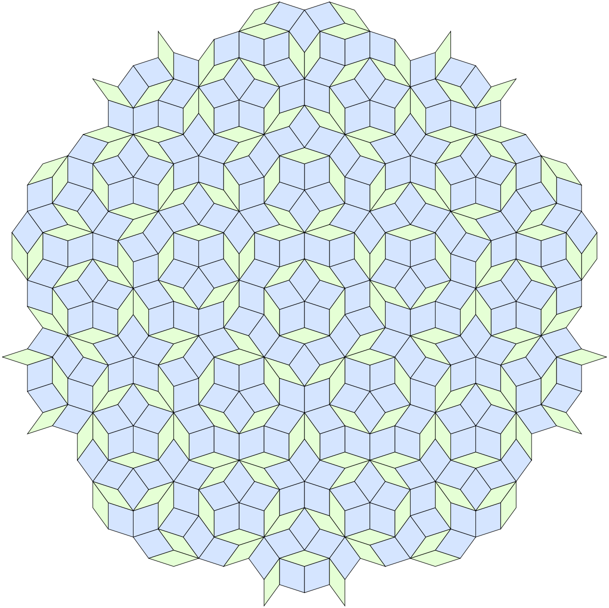
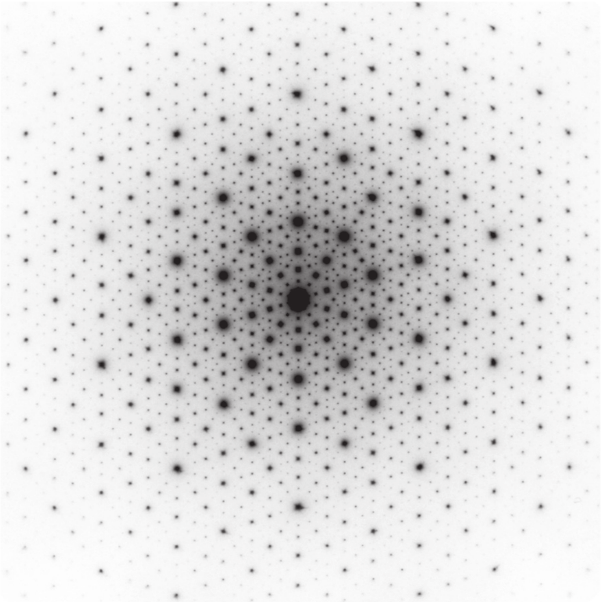
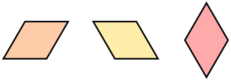


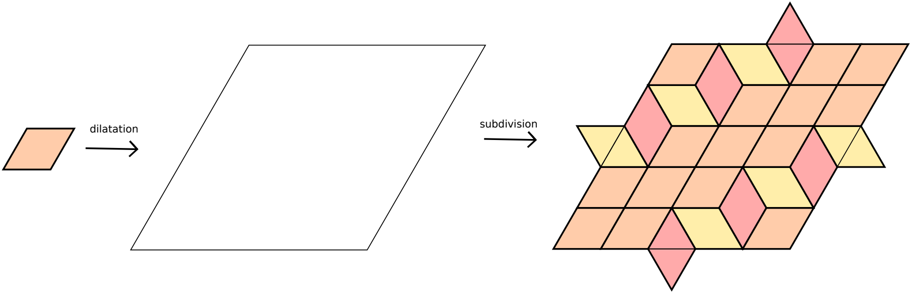


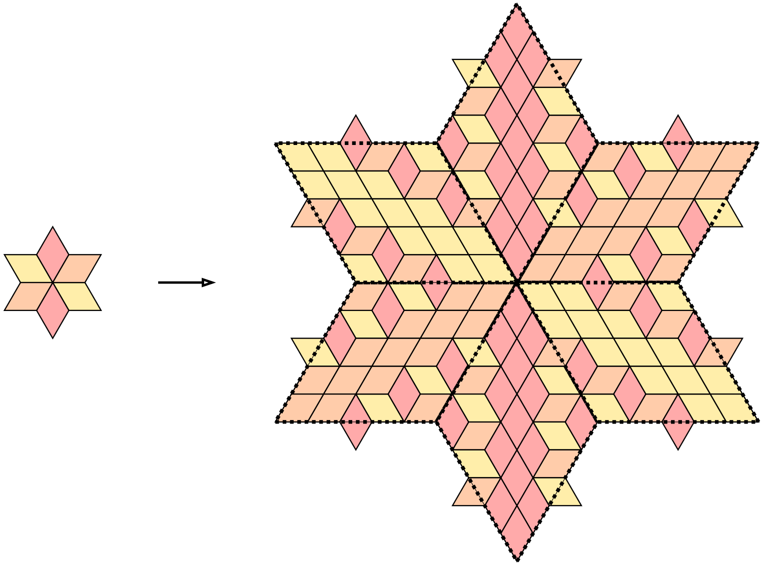
In their 2016 article, Jarkko Kari and Markus Rissanen define a explicit method to construct a substitution tiling with n-fold rotational symmetry for any n. In this talk I will only present the case of $2k + 1$-fold symmetry and first the 7-fold symmetry.

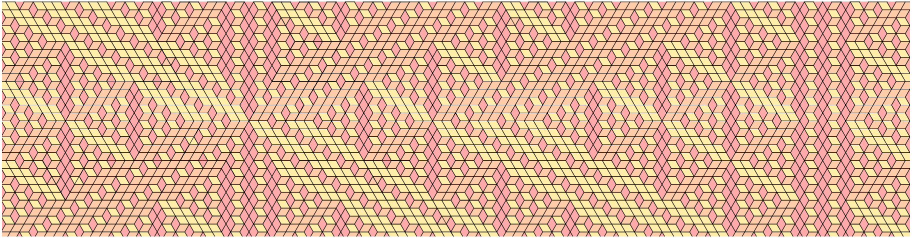
So our first goal was to find a tiling which is:

- defined by a substitution
- cut-and-project
- invariant by rotation of angle $\frac{2\pi}{7}$



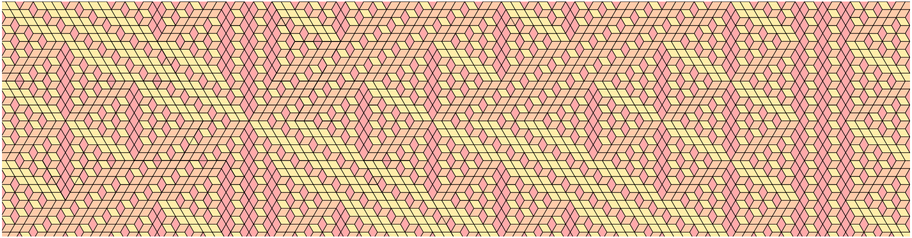

dilatation
subdivision





We have two ways of defining tilings generated by a substitution:

- $\mathcal{T} = \lim_{n \to \infty} \sigma^n(T)$
- The infinitly desubstitutable tilings

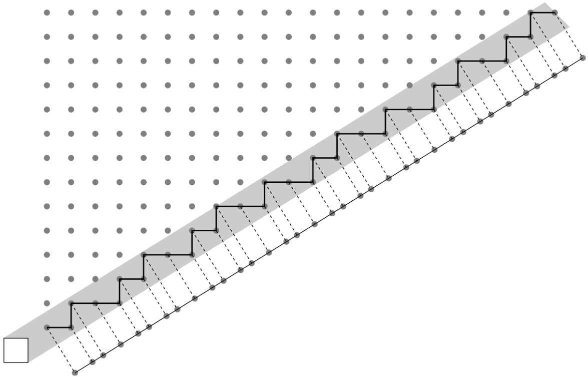


A tiling is *quasiperiodic* when:

- it is not periodic
- it is uniformly recurrent

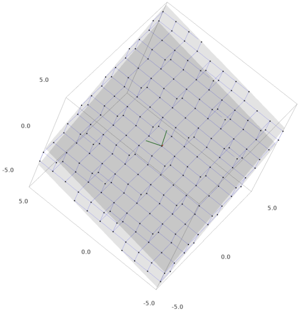
Definition (Uniform recurrence)

A tiling is uniformly recurrent when for every finite patern m that appears in the tiling, there exists a radius r_m so that for every vertex s of the tiling, the pattern m appears at distance $\leqslant r_m$ of s.



Cut $\mathcal{D} + \mathcal{H}$ in grey.
Discrete line $\mathcal{D}_d$:

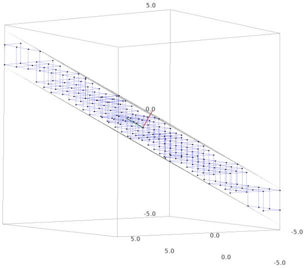
- projection $\mathcal{T}$ on $\mathcal{D}$
- projection Ω on $\mathcal{D}^\perp$.



Setting:

- $\mathbb{R}^n$
- $\mathcal{L} = \mathbb{Z}^n$
- $\mathcal{E}$ irrational plane
- $\mathcal{W}$ and $\mathcal{R}$ orthogonal complementary spaces

Cut $\mathcal{E} + \mathcal{H}$ where $\mathcal{H}$ is a compact set with non empty interior.
Discrete plane $\mathcal{E}_d = \mathcal{L} \cap (\mathcal{E} + \mathcal{H})$
Projection $\mathcal{T} = \Pi_{\mathcal{E}}(\mathcal{E}_d)$ and window $\Omega = \Pi_{\mathcal{W} \oplus \mathcal{R}}(\mathcal{E}_d)$.



$$\mathcal{E} \text{ is generated by } \left(\cos\left(\frac{2k\pi}{7}\right)\right)_{k=0..6} = \begin{pmatrix} 1 \\ \cos\left(\frac{2\pi}{7}\right) \\ \cos\left(\frac{4\pi}{7}\right) \\ \cos\left(\frac{6\pi}{7}\right) \\ \cos\left(\frac{8\pi}{7}\right) \\ \cos\left(\frac{10\pi}{7}\right) \\ \cos\left(\frac{12\pi}{7}\right) \end{pmatrix} \text{ and } \left(\sin\left(\frac{2k\pi}{7}\right)\right)_{k=0..6} = \begin{pmatrix} 0 \\ \sin\left(\frac{2\pi}{7}\right) \\ \sin\left(\frac{4\pi}{7}\right) \\ \sin\left(\frac{6\pi}{7}\right) \\ \sin\left(\frac{8\pi}{7}\right) \\ \sin\left(\frac{10\pi}{7}\right) \\ \sin\left(\frac{12\pi}{7}\right) \end{pmatrix}$$

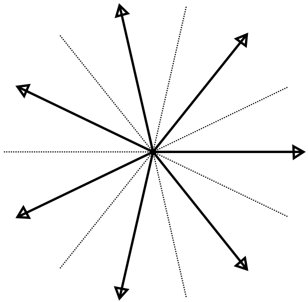


Figure: Projection of the canonical basis $\mathbb{R}^7$

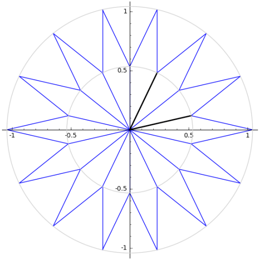
Irrational subspace W: $W = \mathcal{E}' \oplus \mathcal{E}''$ where

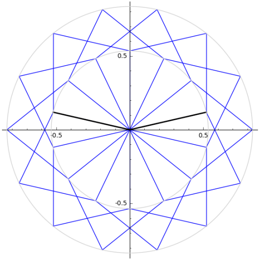
- $\mathcal{E}'$ is generated by $(\cos(\frac{4k\pi}{7}))_{k=0..6}$ and $(\sin(\frac{4k\pi}{7}))_{k=0..6}$
- $\mathcal{E}''$ is generated by $(\cos(\frac{6k\pi}{7}))_{k=0..6}$ and $(\sin(\frac{6k\pi}{7}))_{k=0..6}$

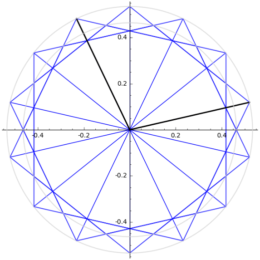
Rational subspace R: $R = \Delta$ line generated by the vector $(1)_{k=0..6}$

$$\boxed{\mathbb{R}^7 = \mathcal{E} \oplus^\perp \mathcal{E}' \oplus^\perp \mathcal{E}'' \oplus^\perp \Delta}$$

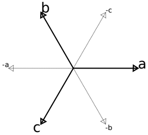
Projections

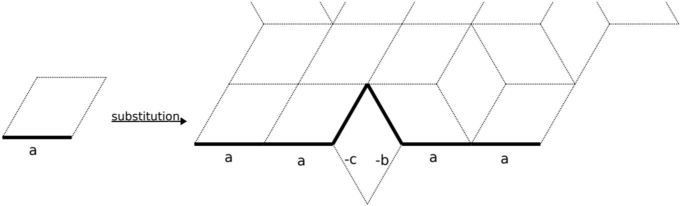






Projection over $\mathcal{E}, \mathcal{E}'$ and $\mathcal{E}''$ of a small set

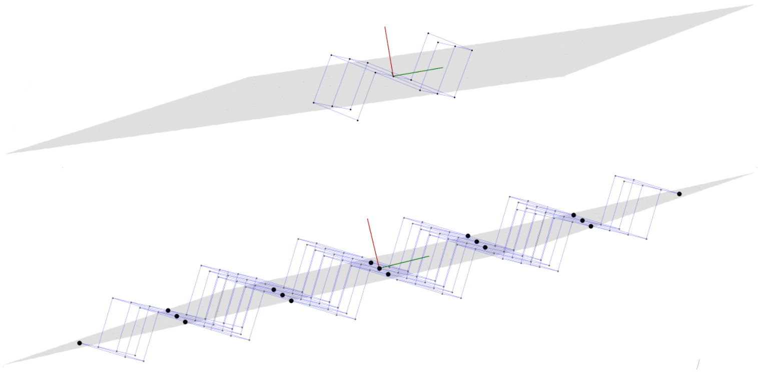




With $a = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$$M = \begin{pmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{pmatrix}$$

We have $\varphi(a) = M \cdot a \ldots$ so φ is described by M.
The study of M leads to a good understanding of the dilatation and substitution "in $\mathbb{R}^3$".



With the setting $\mathbb{R}^n = \mathcal{E} \oplus W \oplus R$ we need:

- $\mathcal{E}$, W, R are eigenspaces
- $|\lambda_{\mathcal{E}}| > 1$
- $|\lambda_W|$, $|\lambda_R| \leqslant 1$

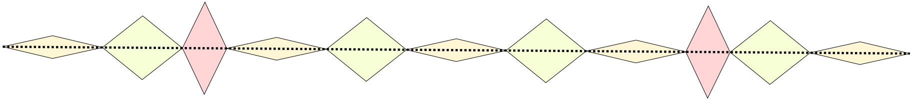
We found

$$C = \begin{pmatrix} 5 & 4 & 0 & -4 & -5 & -2 & 2 \\ 2 & 5 & 4 & 0 & -4 & -5 & -2 \\ -2 & 2 & 5 & 4 & 0 & -4 & -5 \\ -5 & -2 & 2 & 5 & 4 & 0 & -4 \\ -4 & -5 & -2 & 2 & 5 & 4 & 0 \\ 0 & -4 & -5 & -2 & 2 & 5 & 4 \\ 4 & 0 & -4 & -5 & -2 & 2 & 5 \end{pmatrix}$$

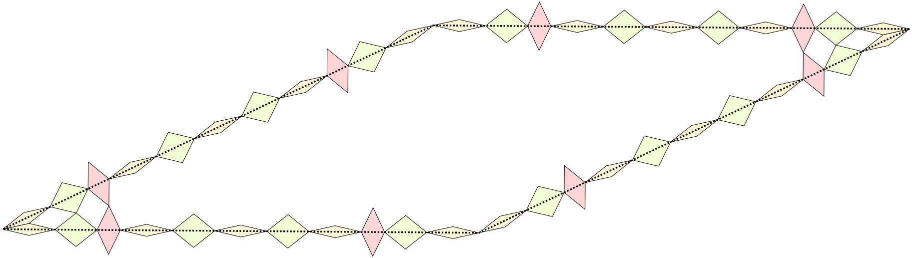
Which has eigenspaces $\mathcal{E}, \mathcal{E}', \mathcal{E}''$ and Δ with eigensvalues

$$|\lambda| \approx 17.7394,\ |\lambda'| \approx 0.4475,\ |\lambda''| \approx 0.3332 \text{ and } \lambda_\Delta = 0$$

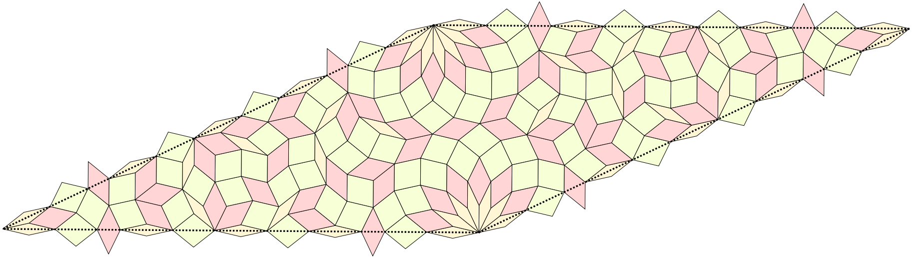
This matrix only defines the edge of the substitution



Tiling the metarhombi



Criterion and tiling algorithm in KENYON93.



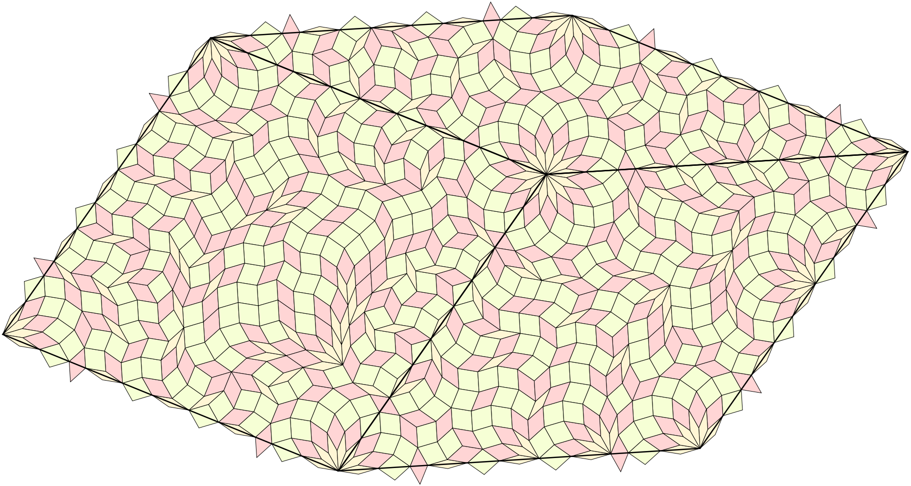
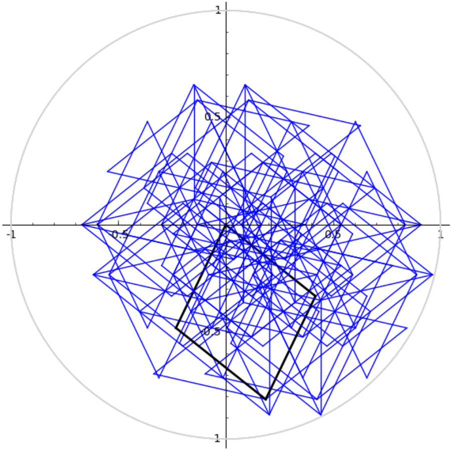
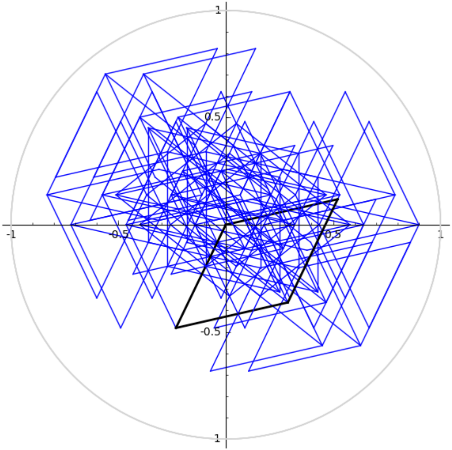


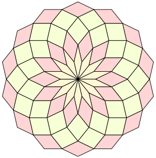
Figure: Complete substitution σ over $\mathcal{E}$

Projections over $\mathcal{E}', \mathcal{E}''$





Main result

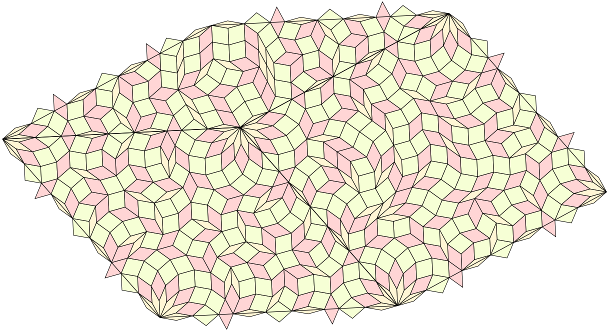


Theorem (Main result)

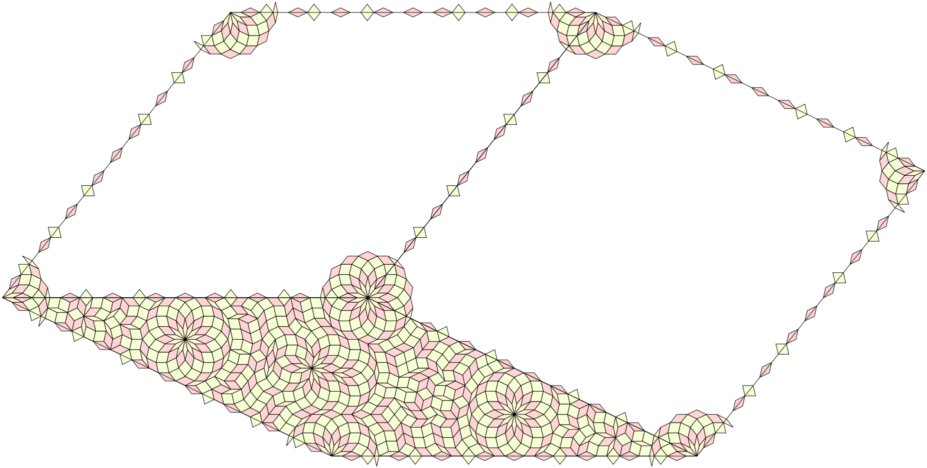
The substitution σ defines a set $\mathcal{E}_d = \lim_{n \to \infty} \sigma^n(R_2^1)$ that satisfies:

- $\mathcal{T} = \Pi_{\mathcal{E}}(\mathcal{E}_d)$ is a rhombus tiling with invariance by rotation of angle $\frac{2\pi}{7}$
- the closure of $\Omega = \Pi_{\mathcal{W} \oplus \mathcal{R}}(\mathcal{E}_d)$ is compact and has a non-empty interior $\Rightarrow$ $\mathcal{E}_d$ is a cut-and-project set.

$$\begin{pmatrix} 4 & 3 & 0 & -3 & -4 & -2 & 2 \\ 2 & 4 & 3 & 0 & -3 & -4 & -2 \\ -2 & 2 & 4 & 3 & 0 & -3 & -4 \\ -4 & -2 & 2 & 4 & 3 & 0 & -3 \\ -3 & -4 & -2 & 2 & 4 & 3 & 0 \\ 0 & -3 & -4 & -2 & 2 & 4 & 3 \\ 3 & 0 & -3 & -4 & -2 & 2 & 4 \end{pmatrix}$$



$$\begin{pmatrix} 10 & 6 & -2 & -9 & -9 & -2 & 6 \\ 6 & 10 & 6 & -2 & -9 & -9 & -2 \\ -2 & 6 & 10 & 6 & -2 & -9 & -9 \\ -9 & -2 & 6 & 10 & 6 & -2 & -9 \\ -9 & -9 & -2 & 6 & 10 & 6 & -2 \\ -2 & -9 & -9 & -2 & 6 & 10 & 6 \\ 6 & -2 & -9 & -9 & -2 & 6 & 10 \end{pmatrix}$$



Cut-and-project setting

The ambiant space is $\mathbb{R}^n$ with $n = 2k + 1$.

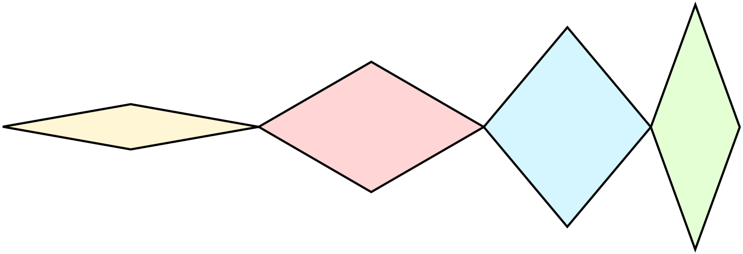
We have $\mathbb{R}^n = \Delta \bigoplus_{i=0..k-1} \mathcal{E}_i$ where

- $\mathcal{E}_i$ is the space generated by the vectors $(\cos(\frac{2(i+1)j\pi}{n}))_{j=0..n-1}$, $(\sin(\frac{2(i+1)j\pi}{n}))_{j=0..n-1}$.
- Δ is the line generated by $(1)_{j=0..n-1}$

$\mathcal{E}_0$ is the tiling plane.

Tiles

We have k rhombus tiles $r_0 \dots r_{k-1}$

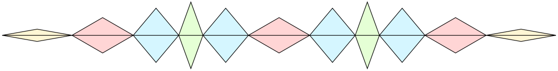


The rhombi r_i has angles $\frac{(2i+1)\pi}{n}$ and $\frac{2(k-i)\pi}{n}$.

So r_0 has narrow angle $\frac{\pi}{n}$ and wide angle $\frac{2k\pi}{n} = \frac{(n-1)\pi}{n}$.

The edges of a substitution will be a sequence of such rhombi $w_1 r_1 + w_2 r_2 + \cdots + w_k r_k$

Example :



Decomposition

For every rhombus r_i we define a dilatation φ_i associated to r_i.

The diagonal vector of r_0 is $e_0 - e_k$ so φ_0 is defined by the matrix $\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & & 0 \\ 0 & 0 & & -1 \\ -1 & 0 & & 0 \\ 0 & -1 & & 0 \\ 0 & 0 & & 0 \\ 0 & 0 & & 0 \\ 0 & 0 & \dots & 1 \end{pmatrix}$

φ_i has $\Delta, \mathcal{E}_j$ for eigenspaces with eigenvalues 0 and $\lambda_{(i,j)} = 2\cos\left(\frac{(2i+1)(2j+1)\pi}{2n}\right)$.

$$\varphi = \sum_{i=0..k-1} w_i \varphi_i$$

Eigenvalues of φ

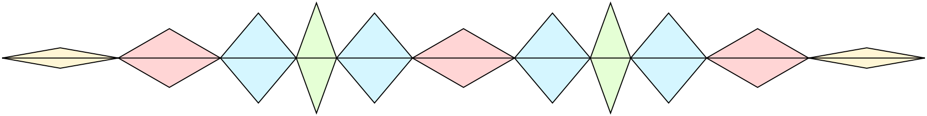
The edges of the substitution are defined by the vector $\begin{pmatrix} w_0 \\ \vdots \\ w_{k-1} \end{pmatrix}$

The dilatation is $\varphi = \sum_{i=0..k-1} w_i \varphi_i$

φ has $\Delta, \mathcal{E}_j$ for eigenspaces with eigenvalues 0 and $\lambda_j = \sum_{i=0..k-1} w_i \lambda_{(i,j)}$

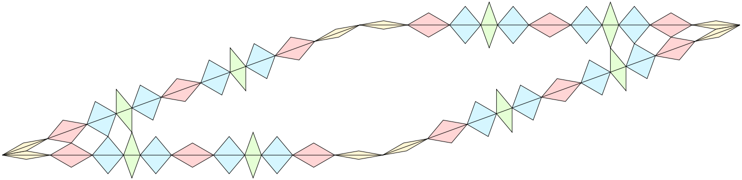
$$\text{So we have } \begin{pmatrix} \lambda_0 \\ \vdots \\ \lambda_{k-1} \end{pmatrix} = \begin{pmatrix} \lambda_{(0,0)} & \cdots & \lambda_{(k-1,0)} \\ \vdots & \ddots & \vdots \\ \lambda_{(0,k-1)} & \cdots & \lambda_{(k-1,k-1)} \end{pmatrix} \cdot \begin{pmatrix} w_0 \\ \vdots \\ w_{k-1} \end{pmatrix}$$

$$\text{With } \lambda_{(i,j)} = 2\cos\left(\frac{(2i+1)(2j+1)\pi}{2n}\right)$$



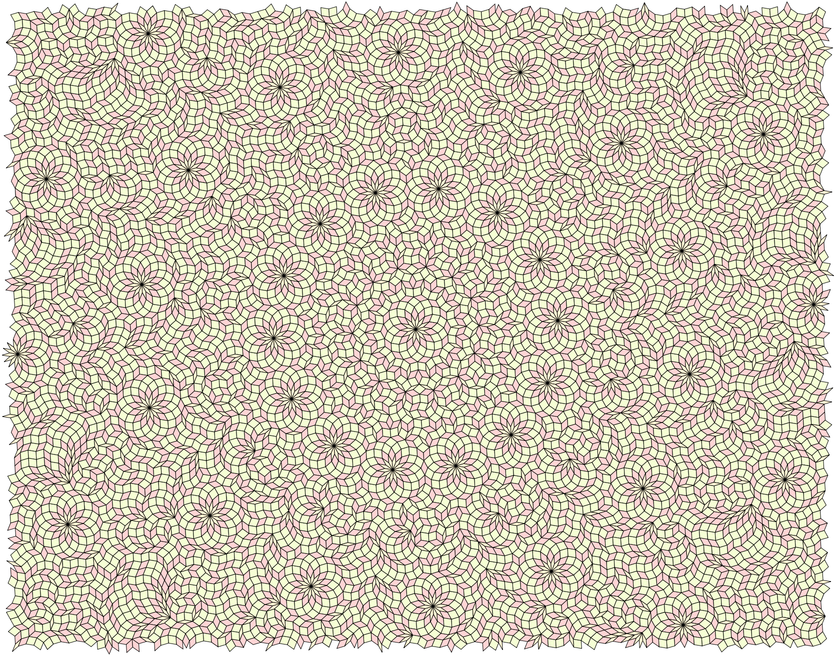
So now we want an edge such that $\begin{pmatrix} w_0 \\ \vdots \\ w_k \end{pmatrix}$ such that:

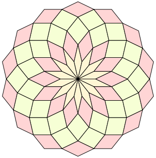
- $|\lambda_0| > 1$ and the other $|\lambda_j| < 1$
  This makes the dilatation admissible for cut-and-project
- the meta-tiles are tilable over $\mathcal{E}_0$
  This makes it an actual substitution of the plane



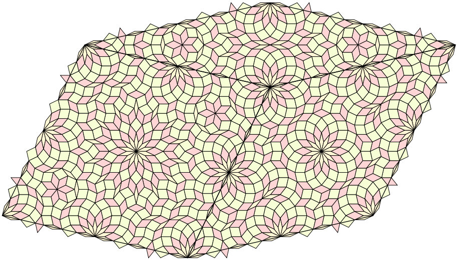
1. The 7-fold case is quite well known now
    - We have 2 explicit substitution 7-fold cut-and-project tilings
    - We have a caracterisation of these tilings
2. We designed the methodology for arbitrary dimension n
    - For any dimension we can easily have the existence of admissble substitution matrices
    - The only thing missing is tilability of such dilated-tiles

So now we need to find a sequence $\left(\begin{pmatrix} w_0 \\ \vdots \\ w_{k-1} \end{pmatrix}\right)_{k\in\mathbb{N}}$ such that for all k it defines a substitution $(2k+1)$-fold cut-and-project tiling.





$$\begin{pmatrix} 6 & 4 & 0 & -4 & -6 & -2 & 2 \\ 2 & 6 & 4 & 0 & -4 & -6 & -2 \\ -2 & 2 & 6 & 4 & 0 & -4 & -6 \\ -6 & -2 & 2 & 6 & 4 & 0 & -4 \\ -4 & -6 & -2 & 2 & 6 & 4 & 0 \\ 0 & -4 & -6 & -2 & 2 & 6 & 4 \\ 4 & 0 & -4 & -6 & -2 & 2 & 6 \end{pmatrix}$$



$$|\lambda_1| = S \approx 19.6893$$
$$|\lambda_2| \approx 2.0112$$
$$|\lambda_3| \approx 0.5345$$
$$|\lambda_4| = 0$$

Let $n \in \mathbb{N}$. Let $\mathbb{R}^n$ with the canonical basis $(e_1, \ldots, e_n)$, the set of canonical vectors $S = \{\pm e_1, \ldots, \pm e_n\}$ and a subspace W. We define Π_W as the orthogonal projection on W.

Definition

We define the property linked over the sets by

$$\text{linked}(X) \Leftrightarrow \left(\forall x, y \in X, \exists k, x_0 \ldots x_k, \begin{cases} x_0 = x \\ x_k = y \\ \forall 0 \leqslant i < 1, \exists \varepsilon \in S = \{\pm e_1, \ldots, \pm e_n\},\ x_{i+1} = x_i + \varepsilon \end{cases} \right)$$

.

We call unit square a set X such that $\exists x \in \mathbb{R}^n, \exists e, e' \in S \text{ with } e \neq -e', X = \{x,\ x+e,\ x+e',\ x+e+e'\}$.

Given a set Y, a family $(X_i)_{i \in I}$ of unit square is called a total covering by unit squares of Y when

$$\begin{cases} \forall x \in Y, \exists i \in I, x \in X_i \\ \forall\ unit\ square\ X \subseteq Y, \exists i \in I, X = X_i \end{cases},$$

such a covering is called *exact* when $\forall i \in I, \forall x \in X_i, x \in Y$.

Lemma

Let φ a function $\mathbb{Z}^n \to \mathbb{Z}n$, σ a function $\mathcal{P}(\mathbb{Z}^n) \to \mathcal{P}(\mathbb{Z}^n)$ and R_0 a finite linked set such that

1. *σ is the substitution associated to dilatation φ:*
   - $\forall x,\ \sigma(\{x\}) = \{\varphi(x)\}$,
   - $\forall X \subseteq Y,\ \sigma(X) \subseteq \sigma(Y)$
   - $\forall X,\ linked(X) \Rightarrow linked(\sigma(X))$
   - $\exists D \in \mathbb{R}^+$, *for any unit square* X, $\sigma(X)$ *has a diametre* $\leqslant D$ *ie:* $\forall x, y \in \sigma(X),\ d(x, y) \leqslant D$
   - $\forall Y$, *for any total covering by unit squares* $X_1, \ldots, X_p$ *of* Y, *the image* $\sigma(Y)$ *is the union of the images of the* X_i*:* $\sigma(Y) \subseteq \bigcup\limits_{i=1\ldots p} \sigma(X_i)$ *and furthermore if the covering is exact* $\sigma(Y) = \bigcup\limits_{i=1\ldots p} \sigma(X_i)$
2. *φ is contracting on W:* $\exists \lambda < 1,\ \forall x \in \mathbb{Z}^n,\ ||\Pi_W(\varphi(x))|| \leqslant \lambda ||\Pi_W(x)||$
3. $R_0 \subseteq \sigma(R_0)$

We have the following:

1. $R_\infty^W = \lim\limits_{i \to \infty} \Pi_W(\sigma^i(R_0))$ *exists*
2. $\exists M \in \mathbb{R}^+,\ \forall x, y \in R_\infty^W,\ ||x - y|| \leqslant M$
3. $0 \in \bar{R_\infty^W}$