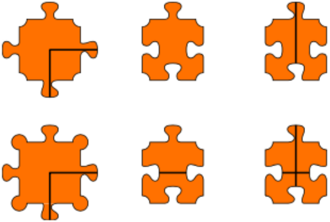
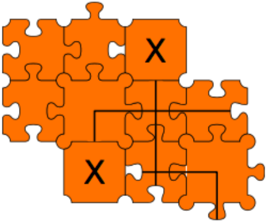
On self-assembly of aperiodic tilings

Galanov Ilya
Laboratoire d'Informatique de Paris Nord
Universite Paris 13

Self-assembly and deceptions





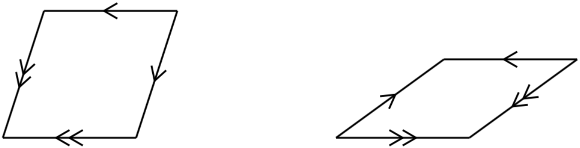
- A set of tiles is called *aperiodic* if copies of them can cover the whole plane (*the tiling*), but only in a non-periodic way.
- Adding tiles one by one may lead to a pattern that cannot be further extended.
- Can we grow a tiling just by adding tiles one by one?

Tiling

By a tiling of $\mathbb{R}^n$ we mean a representation of $\mathbb{R}^n$ as a union of "tiles" where:

- there is a fixed finite set $S = \{p_1, p_2, \ldots, p_l\}$ of "*prototiles*," which are pairwise homeomorphic to the closed ball;
- each tile is an isometric copy of some prototile;
- the interiors of the tiles do not overlap.

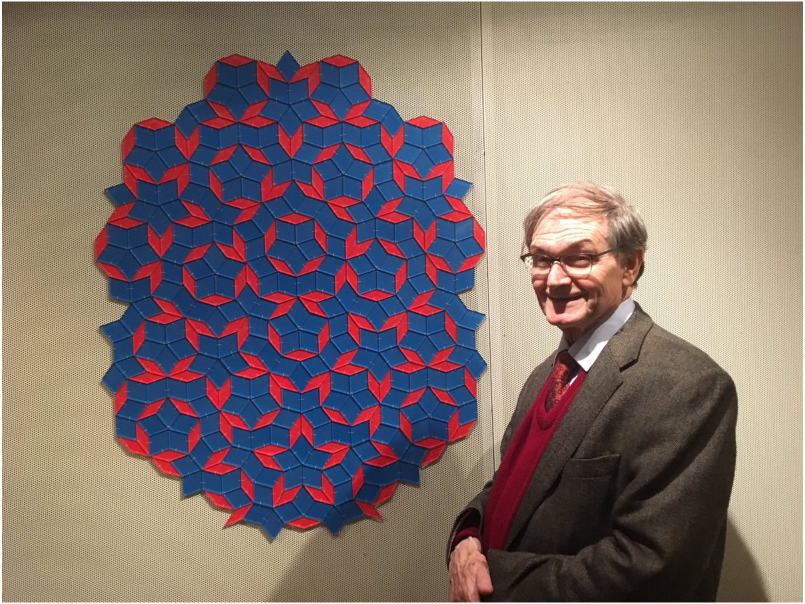
The most famous example: Penrose tiling



Matching rules for Penrose rhombuses.

- The thin rhomb has four corners with angles of 36, 144, 36, and 144 degrees;
- The fat rhomb has angles of 72, 108, 72, and 108 degrees.

The most famous example: Penrose tiling



Deceptions

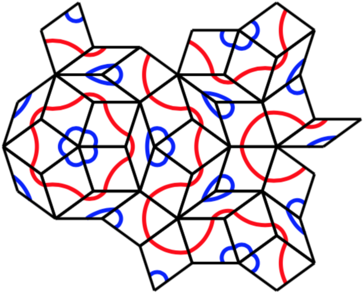
Definition

A patch will be *regular* if it is homeomorphic to the closed ball, and is of *order* r if it covers some disc of radius r.

A regular patch P of order r will be a *deception* of order r if:

- every connected subpatch of P of cardinality less than r is a subset of some tiling of the plane, and
- P is not a subset of any tiling of the plane.

Example



Deception for Penrose tiling.

Deceptions

Theorem (Dworkin-Shieh, 1993)

Any aperiodic protoset admits deceptions of all orders.

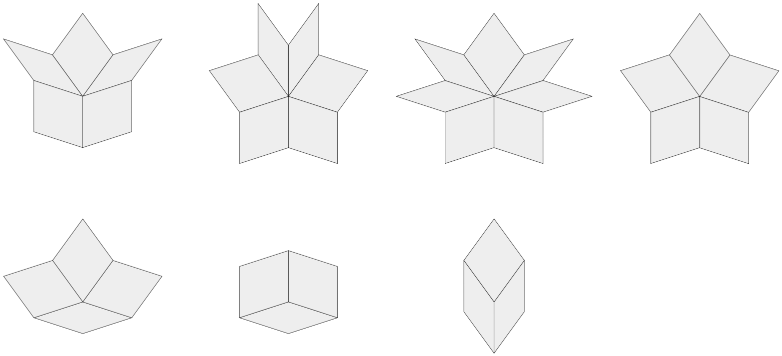
Question

Is it possible to define a *local growth process* that never produces a deception?

Local growth:

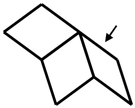
- Tiles must be added one by one;
- Edge must be chosen at random;
- Decision on what tile to add (to do nothing is a valid decision) must be done only by inspecting local configuration around the chosen edge.

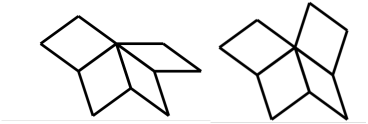
Vertex-atlas



Vertex-atlas for Penrose tiling.

Forced tile example



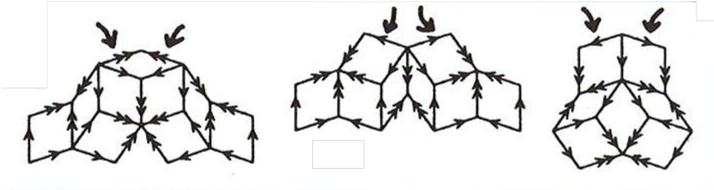


(a) (b)

- (a) is consistent with vertex-atlas;
- (b) is not.

Edge types

- *Forced* – only one tile fits (with respect to vertex-atlas);
- *Unforced* – both fat and thin tiles are possible;
- *Marginal*:



Self-assembly algorithm (Socolar, 1991)

- Start with large enough patch;
- Add tiles to the forced sites until there are none left;
- Add a fat tile to a marginal site;
- Repeat.

Theorem (Socolar, 1991)

Any Penrose tiling can be constructed using described algorithm.

Self-assembly algorithm (Socolar, 1991)

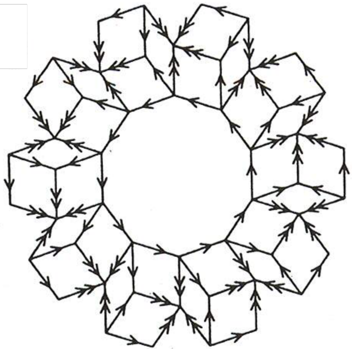
- Start with large enough patch;
- Add tiles to the forced sites until there are none left;
- Add a fat tile to a marginal site;
- Repeat.

Theorem (Socolar, 1991)

Any Penrose tiling can be constructed using described algorithm.

- However, this algorithm is *not* local.

Decapod



Forced growth around this seed will continue ad infinitum.

Canonical cut-and-project

Definition (Planar tiling)

Let E be a d-dimensional affine space in $\mathbb{R}^n$ such that $E \cap \mathbb{Z}^n = \varnothing$. Select the d-dimensional faces with vertices in $\mathbb{Z}^n$ lying in the *strip* $S = E + [0,1]^n$. Project them onto E to get a so-called *planar* $n \to d$ tiling. E is called the *slope* of the tiling.

Definition (Window)

The *window* W of a planar tiling with a slope $E \subset \mathbb{R}^n$ is the orthogonal projection of $[0,1]^n$ onto $E^\perp$, where $E^\perp$ is a complementary space to E:

$$W = \pi^\perp([0,1]^n).$$

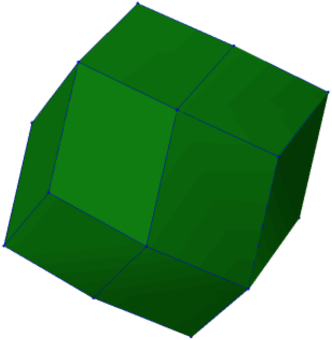
Canonical cut-and-project

Theorem (De Bruijn, 1981)

Penrose tiling is planar $5 \to 2$ *tiling with a slope* E *generated by*

$$u = \begin{pmatrix} 1 \\ \cos(2\pi/5) \\ \cos(4\pi/5) \\ \cos(6\pi/5) \\ \cos(8\pi/5) \end{pmatrix} \quad v = \begin{pmatrix} 0 \\ \sin(2\pi/5) \\ \sin(4\pi/5) \\ \sin(6\pi/5) \\ \sin(8\pi/5) \end{pmatrix}$$

Canonical cut-and-project



Window for Penrose tiling.

Window

To every pattern P we can assign a region in the window:

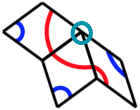
$$R(P) = \bigcap_{x:\pi(x)\in V(P)} (W - \pi^{\perp}(x)).$$

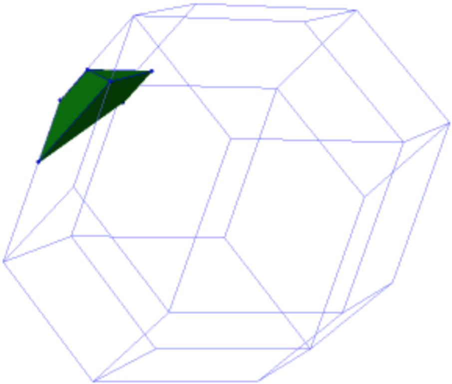
Proposition

In order for pattern P to appear in a tiling it is necessary that

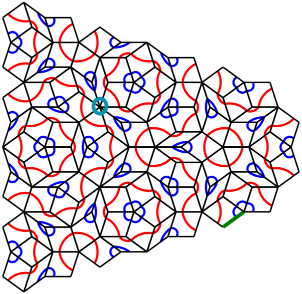
$$R(P) \neq \varnothing.$$

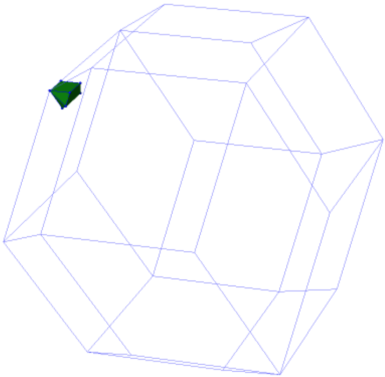
Examples





Examples

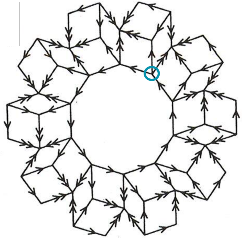


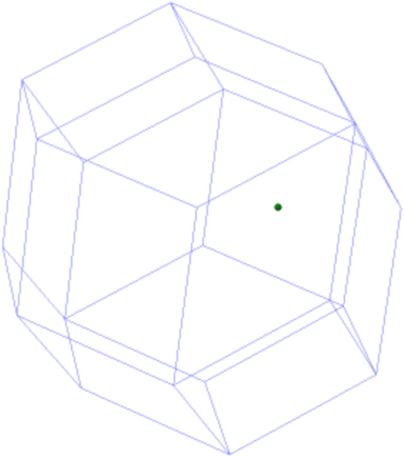


Examples

$$R(Tiling) = \{point\}.$$

Examples





Necessary conditions

Definition (Local rules)

A d-plane $E \subset \mathbb{R}^n$ is said to have local rules if there is a finite set of patterns so that any $n \to d$ tiling without any of these patterns is planar and has a slope parallel to E.

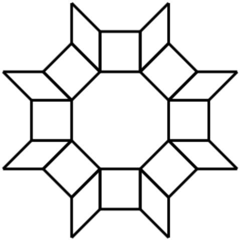
In order to have a deterministic growth self-assembly algorithm for a planar tiling it seems necessary:

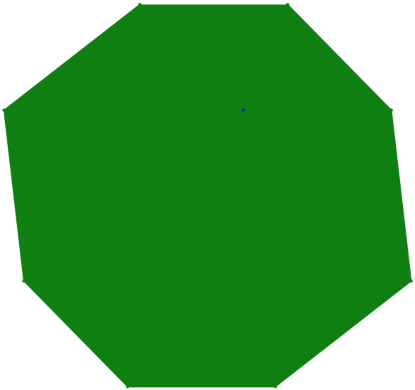
- for slope of the tiling to admit local rules;
- for the starting seed S to satisfy:

$$R(S) = \{point\}.$$

Is it sufficient?

Examples





Thank you for your attention!

Robinson tiling