# Continuous eigenvalues for Meyer sets.

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This is a join work with Daniel Coronel.

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# Introduction.

- Let $D \subset \mathbb{R}^d$ be an aperiodic Delone set and $(X_D, \mathbb{R}^d, \mu)$ be its associated dynamical system with $\mu$ an ergodic measure. In that follows, we denote $X_{D_0}$ the canonical transversal and the groupoid associated by

$$\mathcal{G}_{\mathbb{R}^d} := \{(x,t) \in X_{D_0} \times \mathbb{R}^d / x - t \in X_{D_0}\}.$$

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$$\mathcal{G}_{\mathbb{R}^d} := \{(x,t) \in X_{D_0} \times \mathbb{R}^d / x - t \in X_{D_0}\}.$$

- When $D$ is repetitive, $X_{D_0}$ is a Cantor set and $(X_D, \mathbb{R}^d, \mu)$ is minimal.
- $\alpha \in \mathbb{R}^d$ is an eigenvalue for $(X_D, \mathbb{R}^d, \mu)$ if exists $f \in L^2(X_D, \mu)$ such that for $\mu$-almost every $x \in X_D$ and all $t \in \mathbb{R}^d$ is verifies

$$f(x-t) = e^{2\pi i \langle \alpha, t\rangle} f(x).$$

If $f$ is continuous, then we say that $\alpha$ is a continuous eigenvalue.

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## Theorem

*In $\mathbb{R}^d$, a repetitive Delone set of finite local complexity has $d$ linearly independent continuous eigenvalues if and only if it is topologically conjugate to a Meyer set.*

- In particular, each repetitive Meyer set has $d$ linearly independent continuous eigenvalues.
- We are interested in to find some condition to ensure that all eigenvalues are continuous. Before that, we give a dynamical proof of the fact that a Meyer set has $d$ linearly independent continuous eigenvalues.

# Preliminaries.

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- When the number of generators is $s \geq d$ we say that the rank of $D$ is $s$ and we write $rank(D) = s$.
- Fix a basis $\{v_1, \ldots, v_s\} \subset \mathbb{R}^d$ of $[D]$, i.e. $[D] = \mathbb{Z}[v_1, \ldots, v_s]$. The address map of $D$ is $\phi_D : [D] \to \mathbb{Z}^s$ defined for

$$t = \sum_{i=1}^{s} n_i v_i \quad \text{by} \quad \phi_D(t) = (n_1, \ldots, n_s).$$

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$$t = \sum_{i=1}^{s} n_i v_i \quad \text{by} \quad \phi_D(t) = (n_1, \dots, n_s).$$

- If in addition, $D$ is Meyer then there exists a linear map $L_D : \mathbb{R}^d \to \mathbb{R}^s$ and a constant $\xi_D > 0$ that verifies for all $t \in [D]$,

$$||\phi_D(t) - L_D(t)||_s \leq \xi_D.$$

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- The map $L$ is independent in his first coordinate, i.e. for all $(x,t),(y,t) \in \mathcal{G}_{\mathbb{R}^d}$ we have

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- The map $\Phi - L : \mathcal{G}_{\mathbb{R}^d} \to \mathbb{R}^s$ is a continuous cocycle.

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- If we call $A \in M_{s\times d}(\mathbb{R})$ the matrix, in canonical basis, associated to the linear transformation $L$, then we have the following result.

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**Theorem (A)**

*Let $D \subset \mathbb{R}^d$ be a repetitive Meyer set with $rank(D) = s$. The dynamical system $(X_D, \mathbb{R}^d)$ has $s \geq d$ continuous eigenvalues given by the rows of $A$.*

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## Theorem (A)

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- **Sketch of proof:** For all $x \in X_{D_0}$ consider the fiber

$$\mathcal{G}_{\mathbb{R}^d,x} := \{t \in \mathbb{R}^d : (x,t) \in \mathcal{G}_{\mathbb{R}^d}\}.$$

For all $x \in X_{D_0}$, the set $(\Phi - L)(\mathcal{G}_{\mathbb{R}^d,x})$ is relatively compact.

- Using an extension of the Gottschalk-Hedlund's theorem for groupoids given by J.Renault , exists a continuous function $F : X_{D_0} \to \mathbb{R}^s$ such that

$$\Phi(x,t) - L(x,t) = F \circ r(x,t) - F \circ d(x,t).$$

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- taking exponential in each coordinate, on both sides of the last equality and considering that $\Phi(x,t) = \phi_x(t) \in \mathbb{Z}^s$,

$$\exp(2\pi i \, F_i(x-t)) = \exp(-2\pi i \, \langle A_{i,\cdot}, t\rangle) \, \exp(2\pi i \, F_i(x)),$$

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- So, we can extend the map $f(x) = \exp(2\pi i\, F_i(x))$ to whole the hull for obtain an eigenfunction for the eigenvalue $-A_{i,\cdot}$.
- We conclude that for all $i \in \{1,\dots,s\}$, the vector $-A_{i,\cdot}$ is a continuous eigenvalue for $(X_D, \mathbb{R}^d)$.

■

# Results in one dimension.

- Consider a primitive and recognizable fusion rule

$$\mathcal{F} = \{F_n(j)/1 \leq j \leq J_n\}_{n\in\mathbb{N}}$$

with associated matrices $\{M(n)\}_{n\in\mathbb{N}}$. Denote

$P(n) = M(n)M(n-1)\cdots M(1)$ and $H(n) = (h_n(l) : 1 \leq l \leq J_n)^t$.

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## Proposition

*Let $\mathcal{F}$ be a linearly recurrent, strongly primitive, recognizable fusion rule with FLC. Consider $\mu$ being the unique ergodic measure for $(X_{\mathcal{F}}, \mathbb{R})$. If $\alpha \in \mathbb{R}$ is an eigenvalue of $(X_{\mathcal{F}}, \mathbb{R}, \mu)$, then*

$$\lim_{n \to \infty} \max_{1 \leq k \leq J_n} \left| e^{2\pi i \alpha \cdot h_n(k)} - 1 \right|^2 = 0.$$

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### Theorem (B)

*Consider $D \subset \mathbb{R}$ a linearly repetitive Meyer set such that the associated fusion rule is recognizable. Suppose that for all $m \geq 1$ the heights $h_m(1), \ldots, h_m(J_m)$ are rationally independent. Then $(X_D, \mathbb{R})$ has only continuous eigenvalues.*

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- **Sketch of proof:** Using the previous proposition if $\alpha$ is an eigenvalue, then exist $\tilde{m} \in \mathbb{N}$ and a family of integers $(w_j)_{1 \leq j \leq J_{\tilde{m}}}$ such that

$$\alpha = \sum_{j=1}^{J_{\tilde{m}}} w_j \, \mu_0(C_{F_{\tilde{m}}(j)}).$$

Where $C_{F_n(j)}$ is the set of all tilings in $X_{D_0}$ such that the origin is positioned at the control point of the $n$-tile $F_n(j)$.

- Consider a sequence $\{A_n\}_{n\in\mathbb{N}} \subset X_{D_0}$ defined by $A_n := C_{F_n(1)}$ clearly

$$\cdots \subset A_n \subset \cdots \subset A_2 \subset A_1.$$

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- Since $D$ is a repetitive and Meyer set, the sets $D_n = \{t \in \mathbb{R}/D - t \in A_n\}$ are also repetitive and Meyer sets and verifies

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- So, for all $n > \tilde{m}$ and every $1 \leq j \leq J_n$ exist $t_{1,j}, t_{2,j} \in D_n \subset D_{\tilde{m}}$ with $t_{1,j} > t_{2,j}$ such that $h_n(j) = t_{1,j} - t_{2,j}$ and therefore

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$$\phi_{D_{\tilde{m}}-t_{2,j}}(h_n(j)) = [P_{j,\cdot}(n,\tilde{m})]^t.$$

- For this reason, for all $n > \tilde{m}$ and every $1 \leq j \leq J_n$ we have

$$\left\| \frac{1}{h_n(j)}[P_{j,\cdot}(n,m_0)]^t - \mathcal{L}_{D_{m_0}}(1) \right\|_s \leq \frac{\xi_{D_{m_0}}}{h_n(j)}.$$

- If we denote $\mu(\tilde{m}) = \begin{bmatrix} \mu_0(C_{F_{\tilde{m}}(1)}) \\ \vdots \\ \mu_0(C_{F_{J_{\tilde{m}}}(J_{\tilde{m}})}) \end{bmatrix}$, where $\mu_0$ is the transversal measure, we have

$$\|\mathcal{L}_{D_{\tilde{m}}}(1) - \mu(\tilde{m})\|_s \leq \left\| \frac{[P_{j,\cdot}(n,\tilde{m})]^t}{h_n(j)} - \mathcal{L}_{D_{\tilde{m}}}(1) \right\|_s + \left\| \frac{[P_{j,\cdot}(n,\tilde{m})]^t}{h_n(j)} - \mu(\tilde{m}) \right\|_s$$

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- Taking $n \to \infty$, we have $\mathcal{L}_{D_{\tilde{m}}}(1) = \mu(\tilde{m})$ and by the theorem (A), we conclude that for all $1 \leq i \leq J_{\tilde{m}}$, $\alpha' = \mu_0(C_{F_{\tilde{m}}(i)})$ is a continuous eigenvalue of $(X_{D_{\tilde{m}}}, \mathbb{R})$.

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- Since $(X_{D_{\tilde{m}}}, \mathbb{R})$ is a factor of $(X_D, \mathbb{R})$ we conclude the result.

# Examples 1.

- A sequence $s = (s_i)_{i\in\mathbb{Z}} \in \{0,\dots,m\}^{\mathbb{Z}}$ is **almost linear** if there exist a finite collection of real number $\{\gamma_a\}_{a=0}^{m}$ and some constant $C$ such that the partial sums

$$S_i(a) := \begin{cases} \sum_{k=0}^{i-1} \mathbb{1}_{\{a\}}(s_k) & \text{if } i \geq 1, \\ 0 & \text{if } i = 0, \\ \sum_{k=i}^{-1} \mathbb{1}_{\{a\}}(s_k) & \text{if } i \leq -1 \end{cases}$$

satisfy for all $i \in \mathbb{Z}$, $\max_{0\leq a\leq m}|S_i(a) - i\ sign(i)\ \gamma_a| \leq C$.

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## Theorem (Lagarias, 1999.)

*If $s = (s_i)_{i \in \mathbb{Z}} \in \{0, \ldots, m\}^{\mathbb{Z}}$ is an almost linear sequence, then for all finite collection of rationally independent positive numbers $\alpha_0, \ldots, \alpha_m$ the Delone set $D_{\alpha_0, \ldots, \alpha_m}(s)$ is Meyer.*

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### Theorem

*Let* $s = (s_i)_{i \in \mathbb{Z}} \in \{0, \ldots, m\}^{\mathbb{Z}}$ *be a symbol sequence that is almost linear and such that* $(X_{D_{\alpha_0,\ldots,\alpha_m}(s)}, \mathbb{R})$ *is linearly recurrent. Suppose that the associated fusion rule is recognizable and for all* $n \geq 1$ *the heights* $h_1(n), \ldots, h_{c(n)}(n)$ *are rationally independent, then the transversal system* $(X_{D_{\alpha_0,\ldots,\alpha_m}(s)}, \mathbb{R})$ *has only continuous eigenvalues.*

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### Theorem

*Let $s = (s_i)_{i\in\mathbb{Z}} \in \{0, \dots, m\}^{\mathbb{Z}}$ be a symbol sequence that is almost linear and such that $(X_{D_{\alpha_0,\dots,\alpha_m}(s)}, \mathbb{R})$ is linearly recurrent. Suppose that the associated fusion rule is recognizable and for all $n \geq 1$ the heights $h_1(n), \dots, h_{c(n)}(n)$ are rationally independent, then the transversal system $(X_{D_{\alpha_0,\dots,\alpha_m}(s)}, \mathbb{R})$ has only continuous eigenvalues.*

- Also, using theorem (A), we can observe that for all $1 \leq k \leq m$ the real numbers

$$\beta_0 = \frac{m\gamma_1 - 1}{m\gamma_1(\alpha_0 - \alpha_1) - \alpha_0} \text{ and } \beta_k = \frac{m\gamma_k}{m\gamma_k(\alpha_k - \alpha_0) + \alpha_0},$$

are continuous eigenvalue of $(X_{D_{\alpha_0,\dots,\alpha_m}(s)}, \mathbb{R})$.

# Example 2.

- Define the substitutions $\sigma_A, \sigma_B : \{1,2\} \to \{1,2\}^\star$ by

$$\sigma_A : \begin{cases} \sigma_A(1) = 2211111 \\ \sigma_A(2) = 22211 \end{cases} \quad \text{and} \quad \sigma_B : \begin{cases} \sigma_B(1) = 211 \\ \sigma_B(2) = 21. \end{cases}$$

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- Consider the sequence $(v_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ inductively by $v_1 = 1$ and

$$v_{n+1} = \begin{cases} \beta_A v_n & ;\text{if} \quad n v_n \leq 1 \\ \beta_B v_n & ;\text{if} \quad n v_n > 1. \end{cases}$$

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- Define a fusion rule with tiles of level $n$ defined by the substitutions $\sigma_1 = \mathbb{I}$ and $\sigma_{n+1} = \sigma_n \circ \sigma_{M(n)}$, where

$$M(n+1) = \begin{cases} A & ;\text{if} \quad n v_n \le 1 \\ B & ;\text{if} \quad n v_n > 1. \end{cases}$$

# Example 2.

- Define the substitutions $\sigma_A, \sigma_B : \{1,2\} \to \{1,2\}^\star$ by

$$\sigma_A : \begin{cases} \sigma_A(1) = 2211111 \\ \sigma_A(2) = 22211 \end{cases} \quad \text{and} \quad \sigma_B : \begin{cases} \sigma_B(1) = 211 \\ \sigma_B(2) = 21. \end{cases}$$

- Consider the sequence $(v_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ inductively by $v_1 = 1$ and

$$v_{n+1} = \begin{cases} \beta_A v_n & ;\text{if} \quad n v_n \leq 1 \\ \beta_B v_n & ;\text{if} \quad n v_n > 1. \end{cases}$$

- Define a fusion rule with tiles of level $n$ defined by the substitutions $\sigma_1 = \mathbb{I}$ and $\sigma_{n+1} = \sigma_n \circ \sigma_{M(n)}$, where

$$M(n+1) = \begin{cases} A & ;\text{if} \quad n v_n \leq 1 \\ B & ;\text{if} \quad n v_n > 1. \end{cases}$$

- This fusion rules allows us construct a sequence $\eta \in \{1,2\}^{\mathbb{Z}}$ that is not almost linear.

Associating lenghts to tile the real line we obtain

- If the vector of initial lenghts are $H(1) = \begin{pmatrix} 1 \\ \phi \end{pmatrix}$, the dynamical system $(X_\eta, \mathbb{R})$ has only to $0$ as a continuous eigenvalue and the set of eigenvalues is given by

$$E = \left\{ e^{2\pi i\alpha} \in \mathbb{C};\ \alpha = \left(\frac{1}{2}, \frac{\phi - 1}{2}\right) \cdot A^{-l} w,\ l \geq 0,\ w \in \mathbb{Z}^2 \right\}.$$

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- If the vector of initial lenghts are $H(1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, the dynamical system $(X_\eta, \mathbb{R})$ has eigenvalues given by

$$E = \mathbb{Z} \cup \left\{ e^{2\pi i \alpha} \in \mathbb{C};\ \alpha = \left( \frac{2\phi - 1}{5}, \frac{3 - \phi}{5} \right) \cdot A^{-l} w,\ l \geq 0,\ w \in \mathbb{Z}^2 \right\}.$$

Where the only continuous eigenvalues are $\alpha \in \mathbb{Z}$.

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